MIXED VARIATIONAL PROBLEMS ASSOCIATED WITH STATIONARY VISCOUS INCOMPRESSIBLE FREE BOUNDARY FLOWS

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by

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To My Parents

Declaration of Candidate

I hereby declare that this thesis is my own work and that it has not been submitted for a degree at any other university.

Signed by candidate

C. le Roux

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Summary

A strategy that is often used in the study of capillary free boundary (FB) problems for viscous incompressible flows is the following:

(1) Ignore one of the boundary conditions at the FB and prove that for every chosen position of the FB the resultant problem, here called the auxiliary problem (AP), is well posed.

(2) Establish regularity results for the solution of the AP.

(3) Using (2) and the remaining boundary condition, determine the position of the FB.

We study the existence and uniqueness of the weak solution(s) to the AP, i.e., step (1), under minimal regularity constraints on the data and domain. The analysis is carried out for stationary two-dimensional flows, governed by either the Stokes or Navier-Stokes equations, in the context of four standard examples. A Green's formula is derived which allows the AP to be formulated as a mixed variational problem in which the pressure and normal stress appear as Lagrange multipliers. Existence and uniqueness results are obtained by using the Ladyzhenskaya-Babuska-Brezzi theory for mixed problems. By analogy with step (3), the dependence of the normal stress on the position of the FB is investigated.

Contents

1 Introduction

Fluid flows with free surfaces

Many problems in mechanics and physics can be modelled by differential equations for the unknown functions on domains which are not known *a priori.* The unknown (portion of) the boundary must be determined as part of the solution and is known as a *free boundary* (FB). A boundary value problem of this type in which the FB is independent of time is called a *stationary* FB problem. A *moving* FB problem is an initial boundary value problem in which the FB also depends on time.

Hydrodynamics abounds with FB problems. Usually the FB occurs as a liquid-gas or liquid-solid interface or as the interface between two immiscible fluids, while the velocity and pressure fields of the fluid(s) are the main unknowns in the problem. An especially important class of problems are those in which capillary forces (i.e., forces due to intermolecular interaction having a nonzero resultant at the boundary of the fluid) play a dominant role. Problems of this type appear in a wide variety of coating and material processing technologies including lubrication, electrochemical plating, separation processes, metal- and glass-forming processes and the processing of coating polymers, semiconductors, single crystals and other advanced materials.

The study of fluid motion with stationary or moving FBs is not only of practical significance but involves particularly interesting nonlinear analysis. The mathematical analysis of problems describing flows with FBs was originated more that a century ago by Helmholtz and Kirchhoff who studied steady, irrotational, planar flows of inviscid incompressible fluids. By introducing the velocity potential and stream function as new variables, these assumptions lead to a formulation in terms of the velocity potential that must satisfy the Laplace equation in a variable domain with suitable boundary conditions. Then, by exploiting the fact that it is a streamline, the FB becomes a known curve in the complex velocity potential plane. Using this approach and methods involving the theory of functions of a complex variable, many significant results have been obtained. For instance, R. Gerber (in 1955, 1957) gave, via Leray-Schauder degree theory, existence and uniqueness results for the flow in an open channel. However, under the assumptions above only conservative external body forces can be taken into account, the fluid cannot satisfy the no-slip boundary condition, and all the FB problems involving viscous fluids are excluded.

FB problems involving viscous fluids arise, in for example coating and polymer technology, from the description of viscous jets and drop motions, wave and cavity flows of viscous fluids, thin fluid layer motions, fluid interfaces and solidification fronts. A FB problem of particular engineering importance is that of the behaviour of a viscous liquid partially filling an open container, possibly in a low gravity environment (in spacecraft, for instance) where capillary effects must be taken into account. A reliable model of the flow in such situations is that of a viscous incompressible (VI) fluid governed by the Navier-Stokes equations.

Stationary FB problems for capillary VI fluids began to be studied rigorously only comparatively recently, following advances in the theory of elliptic boundary value problems. Flows in which the FB has no points in common with the surface of the container or body over which the liquid flows, were found to be the most suitable for mathematical analysis. The first successful study of such a problem was carried out by Pukhnachov (1972a) who considered the steady, periodic, two-dimensional motion of a VI fluid in an open channel, the upper surface of which is free while the bottom represents a rectilinear (or wavy) wall on which there is a periodic distribution of regions of inflow and outflow. It is assumed that on the FB there is no normal velocity and no shear stress, and that the normal component of the internal stress is equal to the surface tension (or capillary pressure), which is proportional to the curvature of the FB. The relation between the position of the FB and the values on the FB of the velocity and pressure fields which is introduced by this equilibrium condition allowed Pukhnachov to solve the problem by means of the so-called *splitting method.*

This method consists of two stages : in the first stage the form of the FB, given by a function f, is fixed, i.e., the FB is replaced by a rigid wall allowing perfect slip. For every choice of f, the resulting mixed boundary value problem without the normal stress boundary condition, called the *auxiliary problem*, is solved via a weak solution method. Then, by utilizing results concerning boundary value problems for elliptic systems – often those of Agmon, Douglis and Nirenberg (1964) or Solonnikov and Scadilov (1973) – it is proved that the solution $(\mathbf{v}(f), p(f))$ of the auxiliary problem has the necessary differential properties. The second stage of the method involves substitution of the value of the normal stress obtained from the auxiliary problem into the ignored boundary condition and invertion of the curvature operator, thereby reducing the FB problem to a fixed point equation $f = F(f)$ where F is a smooth nonlinear operator in a certain Banach space. By transformation of the auxiliary problem to a domain independent of f and the use of suitable *a priori* estimates, it is then proved that for sufficiently small rates of flow, F is a contraction and that the FB problem has a solution in Hölder spaces which is locally unique.

In Pukhnachov (1972b) the results obtained via this method were used to obtain a linear theory of surface waves by studying the linear approximation of the solution with respect to a parameter proportional to the magnitude of the external action. Since the splitting method is essentially a contraction mapping argument, it is equivalent to an iterative procedure for determining the position of the FB. Pukhnachov (1975) used this method of successive approximations to prove the existence of a one-parameter family of rolling waves, bifurcating from the Poiseuille-type flow, for the two-dimensional motion of VI fluid in an open channel under the influence of gravity. Osmolovskii (1975) and Ladyzhenskaya and Osmolovskii (1976) also employed the splitting method to study certain axisymmetric and three-dimensional FB problems.

In the papers mentioned above it was assumed that the FB does not intersect the fixed boundaries. Following the experiments of Joseph, Beavers and Fosdick (1973), Sattinger (1976) studied, for the special case in which the contact angle θ is $\pi/2$, the axisymmetric problem of a capillary tube filled with fluid with a rod inserted in the centre. The rod rotates slowly, with constant angular velocity ε, say, and the upper surface of the fluid adopts a shape which balances the forces of gravity, atmospheric pressure, internal stress and surface tension. To construct an exact solution, a *perturbation method* is used in which the solution is expressed in powers of ε. The region occupied by the fluid is mapped onto a domain which remains fixed throughout the perturbation scheme. In this way a system of partial differential equations is obtained which depends on the parameter ε, but which is defined on a fixed domain. By the implicit function theorem in a Banach space the question of the convergence of the perturbation series is reduced to that of obtaining suitable *a priori* estimates for the solution of a certain linear boundary value problem. In order to ensure that the solution of the problem has the necessary degree of regularity at the ridges where the FB meets the walls of the rod and container, it was assumed that $\theta = \pi/2$. The reason for this is that in the case of an intersection of such surfaces, edges (or corner points in the planar case) are formed on which the derivatives of the velocity can have singularities depending on the angle θ of intersection.

Jean (1980) considered the stationary two-dimensional motion (governed by the Stokes equations) of a VI fluid in a channel of finite length. The fluid is forced through a slot at one end of a channel, flows down the open channel and is drawn off through a second slot. By using the splitting method and results by Merigot (1977) on regularity at corners, he proved that the FB problem has a unique solution if $\pi/4 < \theta < 3\pi/4$ and the rate of flow is sufficiently small.

Solonnikov (1980) obtained existence and uniqueness results for the solution of a similar FB problem in which θ can take on an arbitrarily specified value between 0 and π, namely the two-dimensional problem of the motion of a VI fluid partially filling a rectangular container under the influence of sinks and sources in the bottom. The proof is based on the splitting method but is different from that of Jean (1980), in that the need for explicit estimates of the regularity at the corners is circumvented through the use of special Hölder and Sobolev spaces with weighted norms, the weight being equal to some power of the distance to the set of angular points. A different proof of the existence result,based on the implicit function theorem used by Sattinger (1976), with a more detailed analysis of the dependence on θ of the differentiability properties of the solution, is given in Solonnikov (1982).

The only problems described thus far in which the domain of flow is unbounded, are the channel problems studied by Pukhnachov (1972a, 1972b, 1975). However, his method ceases to work when the flow is not periodic, for instance in the case of a nonperiodic, monotone,inclined channel. Socolescu (1978b, 1980) proved, by means of the splitting method approach, the existence of a solution to the FB problem of the steady two-dimensional motion under gravity of a heavy VI fluid in an open channel of infinite length. The bottom of the channel is rectilinear in neighbourhoods of the infinities up-stream and downstream and has a negative slope. The existence of a weak solution of the associated auxiliary problem is proved in Socolescu (1978a) via results from the theory of operators of monotone type. Motivated by recent experimental and numerical studies of W.G. Pritchard, L.R. Scott and S.J. Tavener, Abergel and Bona (1991) considered a similar problem for the case of steady, highly viscous flow over a bottom configuration which possesses some localized, non-uniform structure. Via a fixed point argument which involves the use of an implicit function theorem and special function spaces with weighted norms, they proved the existence of a solution which decays exponentially rapidly to the unperturbed Poisseuille-Nusselt flow away from the local variation in the channel bottom profile.

Most of the studies mentioned above relied on the splitting method approach, which is made possible only by the use of the surface tension/curvature condition in modelling the behaviour of the FB. Bemelmans (1987, 1988) considered stationary FB problems, for example the flow in a drop of VI fluid, in which the FB is not governed by surface tension, but by the force of self-attraction or the continuity of the normal stress. He obtained existence, uniqueness and regularity results for solutions which are perturbations of known static configurations. The basic tool in the proof is a version of the Nash-Moser hard implicit function theorem.

As a final remark on the literature dealing with stationary FB problems we note that the special case of static capillary FB problems (i.e., problems in which the fluid moves like a rigid body), liquid bridges, etc. has been treated extensively by Myshkis *et al* (1987).

The first study of a general three-dimensional moving FB problem for flow governed by the Navier-Stokes equations is due to Solonnikov (1977). He considered VI fluid motion in a bounded domain, the entire boundary of which is free, under the assumption that the capillary forces are negligible (i.e., the surface tension vanishes) so that the FB is governed by the continuity of the stress across it. For this situation he proved, for some small time interval which depends on the initial data (i.e., the initial velocity field and position of the FB), the existence and uniqueness of a solution in Hölder spaces. The outline of his approach is as follows. First the problem is transformed to Lagrangian coordinates, so that the flow domain is fixed in time : it is the initial domain. Then, in what is analytically the most difficult part of the proof, estimates are derived for the solution of the linearized problem in which the effect of the change of coordinates is ignored. Then the correction terms (i.e., the terms arising from the coordinate change) are estimated. Finally, with the aid of the method of successive approximations, the existence of a solution is established. Beale (1981) considered the motion of a VI fluid, again without surface tension, under the influence of gravity in a semi-infinite three-dimensional domain bounded by a fixed bottom and moving upper surface, both of which approach horizontal planes at infinity. By means of the Lagrangian formulation and a contraction mapping argument he proved the small-time existence of a solution in Sobolev spaces for arbitrarily prescribed initial data. Moreover, he showed that a solution exists for any given time interval if the initial state is sufficiently close to the static situation (when the velocity is zero and the FB horizontal).

For the same problem, with surface tension taken into account, Beale (1984) proved that a solution exists for all time, again in a space defined by Sobolev norms, if the initial state is close enough to static equilibrium. Allain (1985) proved, for an arbitrary initial velocity field, the small-time existence of a solution if the initial FB configuration is sufficiently close to horizontal. Finally, in Allain (1987) this result was derived without the restriction on the initial shape of the FB. It would appear that the question of the large-time existence of a solution is still an open problem.

Motivated by the need for quantitative information in practical applications and supported by the results of analytical studies, significant progress has been made in the computational study of capillary FB problems for flows governed by the Navier-Stokes equations. We shall not attempt to give a detailed account of this. A survey containing several examples and references can be found in Cuvelier and Schulkes (1990). The three basic numerical methods used for static and stationary problems are the so-called trial method (which is the numerical equivalent of the splitting method), a Newton-type method, and the total linearization method. An example of the computational study of a moving FB problem is that of Cuvelier (1985). He derived existence and uniqueness results for the solution of a linearized moving FB problem and its numerical approximation by means of a small perturbation approach for situations close to static equilibrium.

Outline of this work

From the selection of papers discussed above it is clear that the stationary capillary FB problems for VI flows have been studied extensively since the 1970's, both experimentally, theoretically and computationally. However, the focus of the mathematical studies have been on establishing the existence of *classical* solutions. As a consequence the results invariably require strong assumptions on the smoothness of the data (i.e., the given fixed boundaries and the velocity prescribed on it). Since the auxiliary problem can be solved by variational methods under very weak regularity conditions of this kind, it would suggest that the FB problem as a whole can be solved, albeit only in some weak sense, via a purely *variational* approach in the case of less regular data. This notion is strengthened by the work of Beale and Allain mentioned earlier, in which moving FB problems were studied completely within the framework of the Sobolev spaces. The aim of this work is to investigate the extent to which this strategy can be carried out for certain two-dimensional stationary FB problems.

In chapter 2 we introduce four standard problems and formulate the corresponding equations and boundary conditions governing the flow. The mathematical basis of the study is established in chapter 3. The auxiliary problem is formulated as a mixed variational problem in which the normal stress on the FB and the pressure field appear as Lagrange multipliers. In chapter 4 we derive existence and uniqueness results for this problem. In chapter 5 we study the continuous dependence on the position of the FB of the Lagrange multiplier associated with the normal stress. We conclude with a brief discussion of the remaining unresolved issues.

2 Free Boundary Problems for Stationary Viscous Incompressible Flows

In this chapter we describe in detail a number of situations involving the motion of fluid in a domain with a partially free boundary. A brief description of the physical setting (section 2.1) is followed by the introduction of the governing equations (section 2.2) and the formulation of the boundary conditions (section 2.3).

2.1 Examples of free surface flows

We shall restrict our attention to problems where variations in one direction are assumed to be negligible, so that the fluid motion is two-dimensional. Therefore, the problems can be described in terms of plane geometry.

For the later analysis it is important to have a precise definition of the concept of regularity of the boundary of a domain. A comprehensive treatment of this can be found in e.g., Grisvard (1985), pp. 4–14. For our purposes the following definition will suffice:

(2.1) Let Ω be an open subset of $\Re^2$. We say that its boundary $\partial\Omega$ *is of class* $C^{m,t}$ (for some integer $m \geq 0$ and $0 < t \leq 1$) if for every $\mathbf{x} \in \partial\Omega$ there exists a neighborhood O of $\mathbf{x}$ in $\Re^2$ and new (local) orthogonal coordinates $\mathbf{y} = (y_1, y_2)$ such that

(a) O is a hypercube in the new coordinates : $O = \{\mathbf{y} \mid -a_i < y_i < a_i,\ i = 1,2\}$;

(b) there exists a $C^{m,t}$ function ϕ defined in $O' = \{y_1 \mid -a_1 < y_1 < a_2\}$ that satisfies:

$$| \phi(y_1) | \leq a_2/2 \ \ \forall\, y_1 \in O', \ \ \Omega \cap O' = \{\mathbf{y} \in O \mid y_2 < \phi(y_1)\},$$
$$\partial\Omega \cap O = \{\mathbf{y} \in O \mid y_2 = \phi(y_1)\}.$$

We shall say that $\partial\Omega$ is *Lipschitz continuous* when it is of class $C^{0,1}$.

Here ϕ *is of class* $C^{m,t}$ if it belongs to the *Hölder space* $C^{m,t}(O')$ which is defined as follows:

Let $C^0(O')$ denote the space of continuous functions defined in O' and let $C^m(O') = \left\{f \in C^0(O') \mid f^{(\alpha)} \in C^0(O') \, \forall \, 0 \leq \alpha \leq m\right\}$, where $f^{(\alpha)}$ denotes the α-th derivative of f. Let $C^m(\overline{O'}) = \left\{f \in C^m(O') \mid f^{(\alpha)}\right.$ are bounded and uniformly continuous on $O' \, \forall \, 0 \leq \alpha \leq m\}$. Then $C^{m,r}(O')$ is defined as the subspace of $C^m(\overline{O'})$ consisting of those functions f whose derivatives $f^{(\alpha)}$ satisfy, for all $0 \leq \alpha \leq m$, the Hölder condition with exponent t :

There is a constant K_α such that $| f^{(\alpha)}(x) - f^{(\alpha)}(y) | \leq K_\alpha | x - y |^t \, \forall \, x, y \in O'$.

$C^m(\overline{O'})$ and $C^{m,t}(O')$ are Banach spaces with the respective norms

$$\|f\|_{m,\overline{O'}} = \max_{0 \leq \alpha \leq m} \left\{ \sup_{x \in O'} | f^{(\alpha)}(x) | \right\}, \quad \|f\|_{m,t,O'} = \|f\|_{m,\overline{O'}} + \max_{0 \leq \alpha \leq m} \left\{ \sup_{\substack{x, y \in O \\ x \neq y}} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|}{|x-y|^t} \right\}.$$

For further details see, e.g., Adams (1975). □

Let Ω be a bounded domain (i.e., open and connected) in $\Re^2$ with Lipschitz continuous boundary $\partial\Omega$ consisting of three mutually disjoint open manifolds Γ, Σ and Λ such that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma} \cup \bar{\Lambda}, \text{meas}(\Gamma) > 0, \text{meas}(\Sigma) > 0$ and $\bar{\Gamma} \cap \bar{\Sigma} = \phi$. Greater smoothness will be assigned to portions of the boundary when the need arises.

Here Ω represents the flow region, i.e., the space occupied by the liquid, Γ is the *a priori* unknown (or free) part of the boundary, Σ represents those parts of the boundary where the velocity is prescribed, and Λ represents the remaining parts of the boundary (where mixed boundary conditions will be applied, for instance). We shall only consider situations in which the flow is assumed stationary (time-independent or steady). In particular, we shall focus on the following set of problems.

I. Consider the steady motion of a fluid in the annular space between a solid cylindrical surface Σ rotating with constant angular velocity ω and a free boundary Γ on which the pressure is prescribed as a function of the polar angle. The only body force assumed present is gravity, acting in a fixed direction perpendicular to the symmetry axis of the cylinder.

To be precise, let (x_1, x_2) be the components of $\mathbf{x}$ with respect to a cartesian coordinate system such that the direction of gravitational acceleration is $(0, -1)$ and let (θ, r) be the corresponding polar coordinates, i.e., $x_1 = r cos\theta$, $x_2 = r sin\theta$, etc. Then we set $\Sigma = \{\mathbf{x} \in \Re^2 \mid |\mathbf{x}| = 1\}$ (the unit circle), $\Lambda = \phi$ and we assume that Γ has the representation $\Gamma = \{\mathbf{x} \in \Re^2 \mid r = f(\theta),\ 0 \leq \theta \leq 2\pi\}$, where $f : \Re \to \Re$ is an *a priori* unknown 2π - periodic function such that $0 < r_0 \leq f(\theta) \leq r_1 < 1 \ \forall\ \theta \in \Re$, for chosen constants r_0 and r_1. As in Figure 1, $\Omega = \{\mathbf{x} \in \Re^2 \mid f(\theta) < r < 1,\ 0 \leq \theta \leq 2\pi\}$.

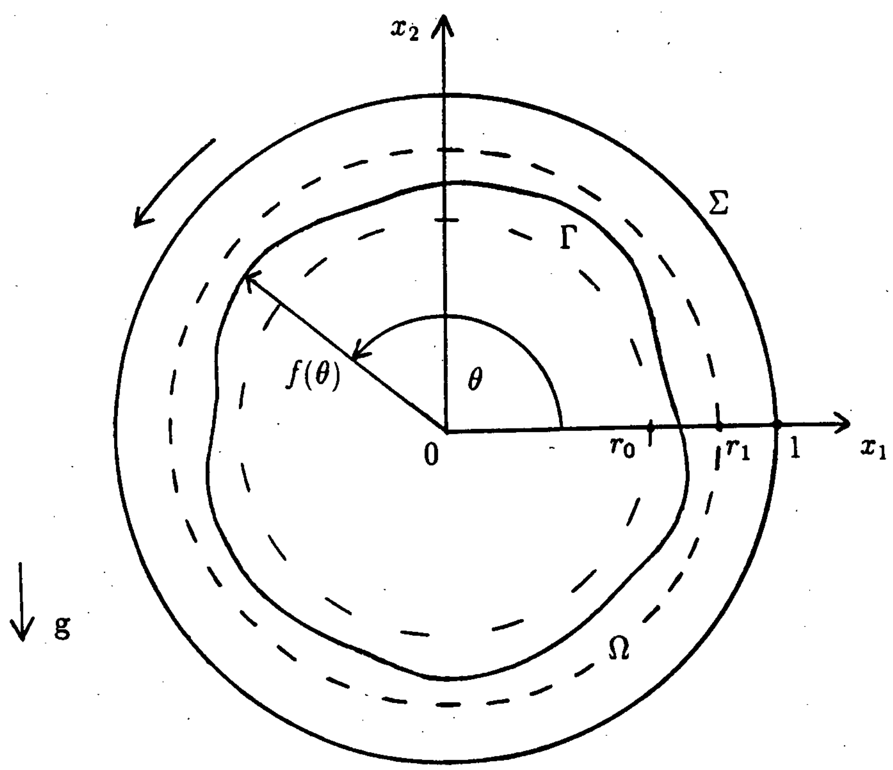


Figure 1. The cylinder problem

II. Here we consider the motion of fluid filling an unbounded curvilinear channel Ω', the upper boundary Γ' of which is free, while the bottom Σ' represents a solid wall with periodically varying shape on which there are periodically distributed regions of fluid ingress and egress. It is assumed that the mentioned periods coincide (and equal 1), that the total intensity of the sinks and sources is zero, and that gravity is present.

For simplicity we assume that Σ' is of the form $\Sigma' = \{\mathbf{x} \in \Re^2 \mid x_2 = b(x_1),\ x_1 \in \Re\}$, with $b : \Re \to \Re$ a given function such that $b(x+1) = b(x)\ \forall\ x \in \Re$, $\int_0^1 b(x)dx = -d$ and $b(x) \leq \delta' - d\ \forall\ x \in \Re$ (e.g., $b(x) \equiv -d$), for given constants $0 < \delta' < d$. Moreover, we assume that Γ' can be given the representation $\Gamma' = \{\mathbf{x} \in \Re^2 \mid x_2 = f(x_1),\ x_1 \in \Re\}$, where $f : \Re \to \Re$ is a (as yet unknown) function such that $\mid f(x) \mid \leq \delta\ \forall\ x \in \Re$, for a given $0 < \delta < d - \delta'$. Thus $\Omega' = \{\mathbf{x} \in \Re^2 \mid b(x_1) < x_2 < f(x_1),\ x_1 \in \Re\}$.

Since the data (Σ' and its distribution of sinks and sources) is 1-periodic in x_1 we can assume that the flow (and hence f) is too. Therefore the problem can be reduced to one on a bounded domain. Let $\Phi = \{\mathbf{x} \in \Re^2 \mid 0 < x_1 < 1\}$ and define $\Gamma = \Phi \cap \Gamma'$, $\Sigma = \Phi \cap \Sigma'$, $\Omega = \Phi \cap \Omega'$ and $\Lambda = \Lambda_0 \cup \Lambda_1$ where $\Lambda_x = \{\mathbf{x} \in \Re^2 \mid x_1 = x,\ b(x) < x_2 < f(x)\}$. See Figure 2.

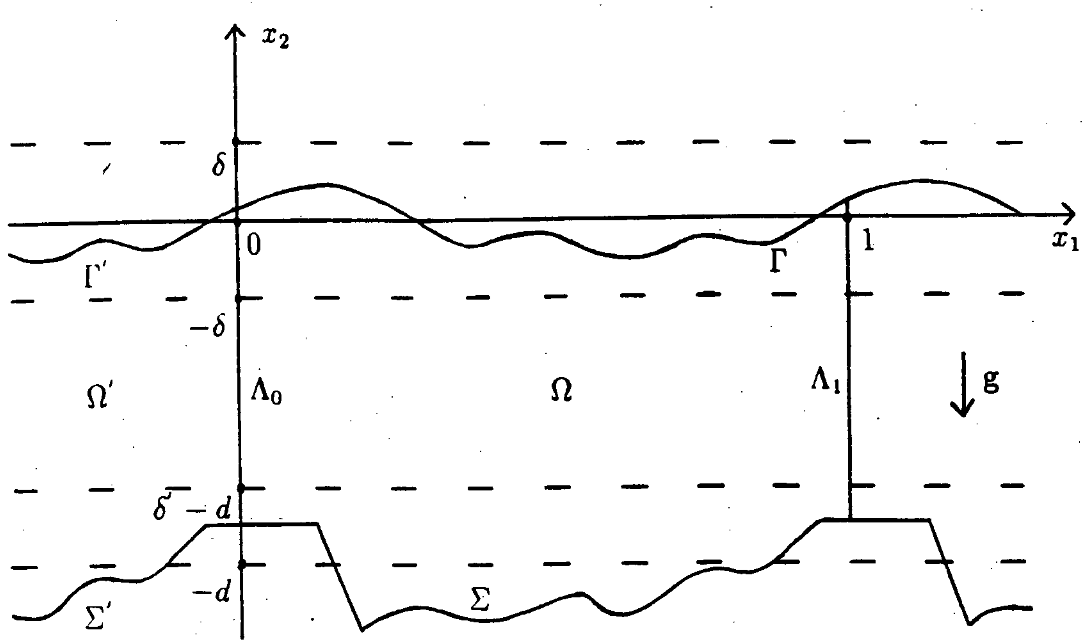


Figure 2. The periodic channel problem

Problems I and II were first studied (for the case of zero gravity) by Pukhnachov (1972a). They have the important feature that the free surface of the liquid does not intersect the surface of the container or the body over which the liquid flows. Furthermore, the flows are both periodic in some sense.

III. The extrusion of a viscous incompressible jet from a die (or nozzle) into an inviscid fluid constitutes one of the best known free boundary problems. It is referred to as the die-swell problem since it is observed that far downstream the height of the extrudate is different from that of the die. We shall assume the conventional geometry for the problem, i.e., we impose *a priori* the condition that the free surface should separate from the boundary at a specified point, such as a sharp lip (cf. Kruyt *et al* (1988) or Cuvelier and Schulkes (1990)). (Note however that the experiments of Jean and Pritchard (1980) indicate that this is in general not a valid assumption.) Moreover, it is assumed that there are no external forces.

The situation is as in Figure 3. In the absence of body forces we can assume that the flow is symmetric with respect to the x_1- axis and thus restrict the analysis to the half-plane $\{\mathbf{x} \in \Re^2 \mid x_2 \geq 0\}$. The free surface is represented by $\Gamma = \{\mathbf{x} \in \Re^2 \mid x_2 = f(x_1),\ 0 < x_1 < 1\}$, where the (unknown) function f must be such that $f(0) = b$ (separation at A) and $f'(1) = 0$ (vanishing slope at B). Furthermore, $\Sigma = \{\mathbf{x} \in \Re^2 \mid x_1 = -a,\ 0 < x_2 < b\}$ (with $a, b > 0$ given constants) is the inlet, $\Lambda_w = \{\mathbf{x} \in \Re^2 \mid -a < x_1 < 0,\ x_2 = b\}$ is the wet part of the wall, $\Lambda_s = \{\mathbf{x} \in \Re^2 \mid -a < x_1 < 1,\ x_2 = 0\}$ represents the symmetry line, $\Lambda_o = \{\mathbf{x} \in \Re^2 \mid x_1 = 1,\ 0 < x_2 < f(1)\}$ is the outlet, and Ω is the bounded domain with boundary $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma} \cup \bar{\Lambda}$ where $\Lambda = \Lambda_w \cup \Lambda_s \cup \Lambda_o$.

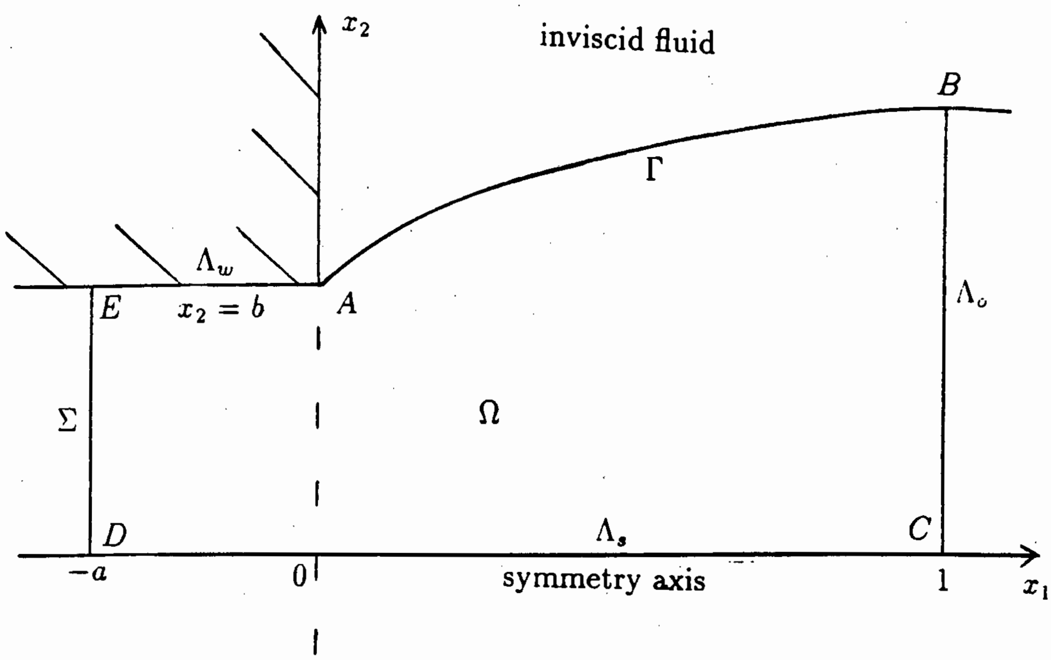


Figure 3. The die-swell problem

IV. A problem of special interest concerns the behaviour of fluids partially filling an open <u>container</u> in a low-gravity environment, where capillary free boundaries must be taken into account. The steady motion is due to sinks and sources of total intensity zero distributed over a portion of the container wall. This problem (under varying geometric conditions) has attracted a significant amount of research (cf., e.g., Solonnikov (1980, 1982), Jean (1980), Pukhnachov (1982), Cuvelier and Schulkes (1990)).

Consider a container Λ' consisting in part of two parallel rectilinear walls, say $\Lambda_0' = \{\mathbf{x} \in \Re^2 \mid x_1 = 0,\ x_2 > -h\}$ and $\Lambda_1' = \{\mathbf{x} \in \Re^2 \mid x_1 = 1,\ x_2 > -h\}$ for some $h > 0$, the lower endpoints of which are connected by a curve Λ_l' satisfying the condition $x_2 \leq -h \ \ \forall\ \mathbf{x} \in \Lambda_l'$. It is assumed that the free boundary $\bar{\Gamma}$ has a single point in common with each of Λ_0' and Λ_1' and does not intersect Λ' at any other point. Moreover, it is assumed that Γ is of the form $\Gamma = \{\mathbf{x} \in \Re^2 \mid x_2 = f(x_1),\ 0 < x_1 < 1\}$, where the (unknown) function $f : [0,1] \rightarrow \Re$ is such that $\mid f(x) \mid \leq \delta < h \ \forall\, x \in [0,1]$ for a given $\delta > 0$.

As before, Σ denotes those parts of the boundary where the flow is prescribed and Λ represents the remainder of the wet part of the container wall. We assume that $\Sigma \subset \Lambda_l'$ and that the vessel is tilted in such a way that the angle between the negative x_2-axis and the direction of gravity equals θ_0. Define $\Lambda_0 = \Lambda \cap \Lambda_0'$, $\Lambda_1 = \Lambda \cap \Lambda_1'$ and $\Lambda_l = \Lambda \cap \Lambda_l'$. The domain Ω is as in Figure 4.

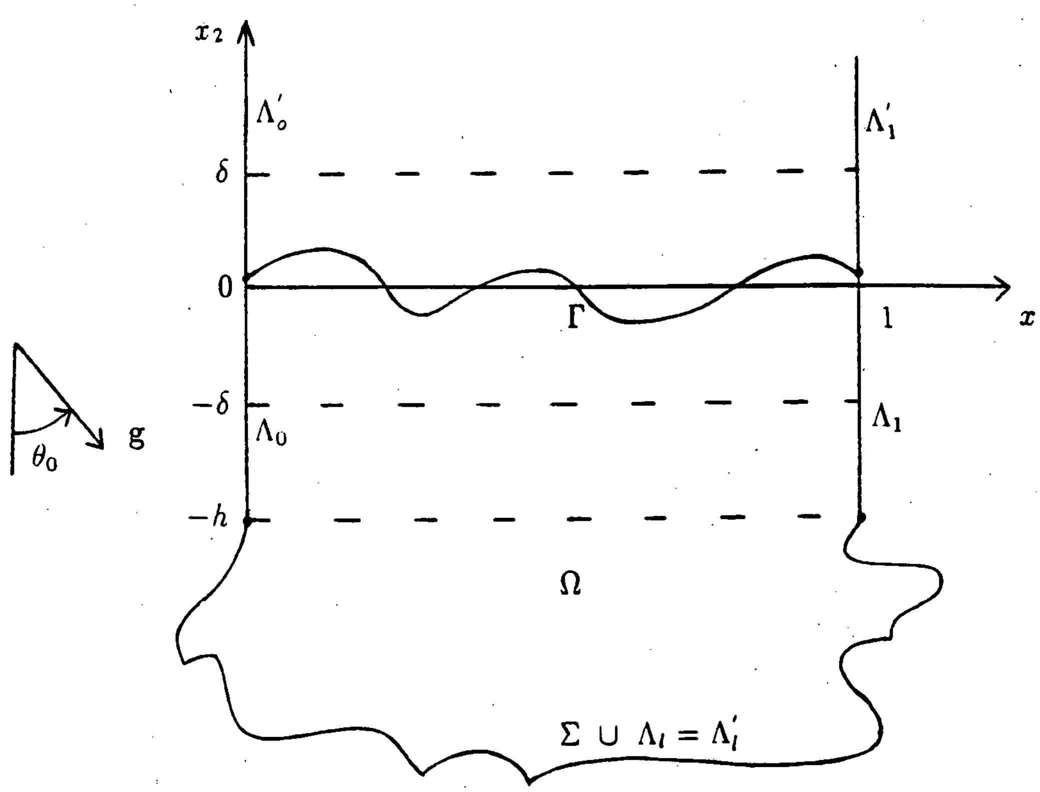


Figure 4. The container problem

Figure 5 illustrates an example considered by Jean (1980). The fluid is forced through a slot at one end of a finite channel, flows down the channel, and is drawn off through a second slot. Here $\Sigma = \Sigma_i \cup \Sigma_o$ and $\Lambda_l = \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$.

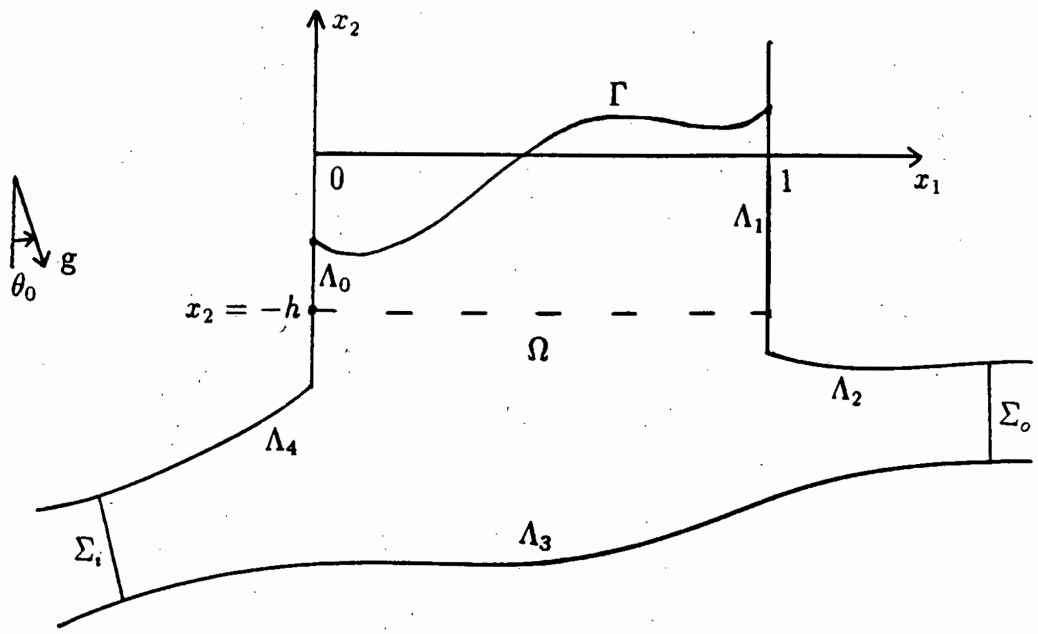


Figure 5. An example of the container problem

In each of the situations I - IV, the problem is to find the position of the free surface and corresponding velocity and pressure fields which satisfy the equations of motion and boundary conditions (to be given in sections 3.2 and 3.3, respectively).

Note that the regularity requirements imposed on $\partial\Omega$ imply corresponding smoothness conditions for the functions f (and b) appearing in problems I - IV. In the problem descriptions above we have tacitly assumed that these conditions are satisfied.

It is instructive to compare the hydrodynamical free surface problems with the so-called Signorini problem in the theory of elasticity. Here one considers the unilateral contact of a body $\bar{\Omega}$ of linearly elastic material with a rigid frictionless foundation F. The body is subjected to surface tractions applied to a portion Λ of its boundary $\partial\Omega$ as well as body forces. The body is fixed along a portion Σ of its boundary and we denote by Γ a portion of the body which is a candidate contact surface, i.e., the actual surface which comes in contact with F is not known in advance but is contained in Γ. See Figure 6.

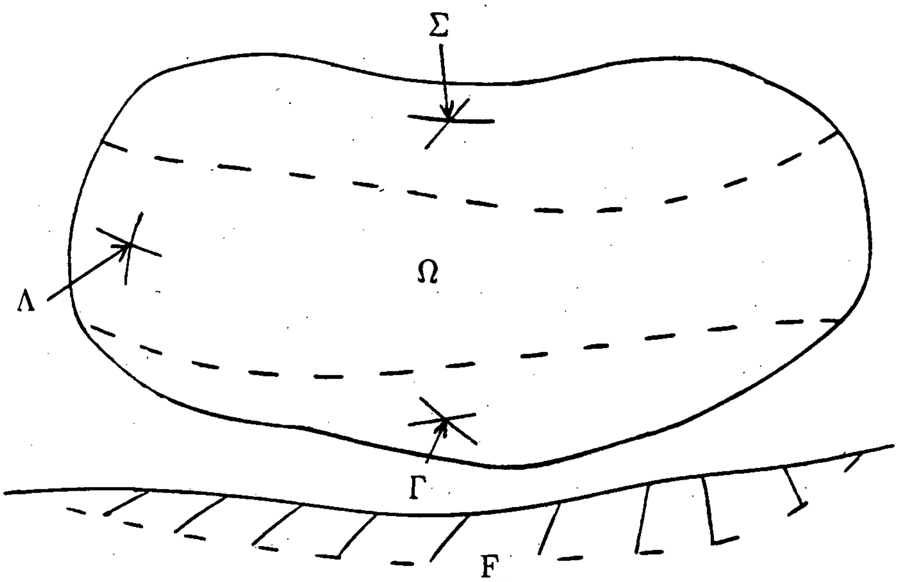


Figure 6. The Signorini problem

With these definitions, the general geometric constraints given at the beginning of this section are precisely those (for either two - or three dimensional models) under which the well-posedness of the Signorini problem (for the case of isotropic incompressible material) can be proved (cf. Kikuchi and Oden (1988), chapter 7). It will be seen later that the link between the two classes of problems is not merely geometrical but also analytical.

2.2 Equations of motion

We assume that the fluid is isotropic, homogeneous, incompressible, viscous and Newtonian. Moreover, we have assumed that the flow is two-dimensional, stationary and isothermal. Under these assumptions the governing equations of the problem, which are the equations for conservation of momentum and mass and the constitutive equation, reduce to the well-known Navier-Stokes (NS) equations for an incompressible fluid:

(2.2) $\rho u_j u_{i,j} - T_{ij,j} = F_i,\ i = 1,2$ (momentum equations),

(2.3) $u_{i,i} = 0$ (incompressibility condition),

where $T_{ij}(\mathbf{u}, P) = -P\delta_{ij} + 2\mu D_{ij}(\mathbf{u})$ and $D_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ 1 \le i,j \le 2$ (constitutive equation).

Here $\mathbf{u} = (u_1, u_2)$ denotes the velocity of the fluid, P is its pressure, $T = [T_{ij}]$ is the stress tensor and $\mathbf{F} = (F_1, F_2)$ represents the external forces. The positive constants ρ and μ are the density and viscosity of the fluid, respectively. Moreover, $(\cdot)_{,k}$ denotes $\frac{\partial(\cdot)}{\partial x k}$, $k = 1, 2$, and the usual summation convention is used, i.e., when an index, say i, appears twice in a single term, then the symbol $\sum_{i=1}^{2}$ will be understood to precede it.

When (2.3) holds, we have the identity

(2.4) $2D_{ij}(\mathbf{u})_{,j} = u_{i,jj} + u_{j,ij} = u_{i,jj} + (u_{j,j})_{,i} = u_{i,jj},\ i = 1, 2.$

Set $p = P/\rho$, $\nu = \mu/\rho$ and $f_i = F_i/\rho$, $i = 1, 2$. Here, p is the kinematic pressure, ν is the kinematic viscosity and $\mathbf{f} = (f_1, f_2)$ represents a density of body forces per unit mass. Using these definitions and (2.4), the NS equations become

(2.5) $u_j u_{i,j} - \nu u_{i,jj} + p,i = f_i,\ i = 1,2,$

(2.6) $u_{i,i} = 0.$

In order to illuminate the relative effect of the viscosity, density, domain size, etc., we now derive a nondimensional form of the NS equations. For a given problem, let L be a characteristic (reference) length and U a characteristic velocity. Then $T = L/U$ is a characteristic time. Define dimensionless variables by

(2.7) $\mathbf{x}^* = \mathbf{x}/L,\ t^* = t/T,\ \mathbf{u}^*(\mathbf{x}^*) = \mathbf{u}(\mathbf{x})/U,\ p^*(\mathbf{x}^*) = p(\mathbf{x})/U^2$ and $\mathbf{f}^*(\mathbf{x}^*) = L\mathbf{f}(\mathbf{x})/U^2$.

The Reynolds number is the dimensionless constant $Re = LU/\nu$. With this change of variables the NS equations assume the form

(2.8) $u_j^* u_{i,j}^* - S_{ij,j}^* = f_i^*,\ i = 1, 2,$

(2.9) $u_{i,i}^* = 0,$

where $S_{ij}^*(\mathbf{u}^*, p^*) = -p^*\delta_{ij} + \frac{2}{Re} D_{ij}^*(\mathbf{u}^*)$, $D_{ij}^*(\mathbf{u}^*) = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*)$ and the differentiation is with respect to $\mathbf{x}^*$.

Henceforth we shall work exclusively with this form of the NS equations and denote the dimensionless variables without asterisks. (Note that this notation was already used in the previous section. The domain Ω, free boundary Γ, etc. introduced there, represent configurations in the $\mathbf{x}^*$ - plane. This explains the presence of a line segment of (dimensionless) length 1 in each of Figures 1 - 5; these segments correspond to ones of length L in the $\mathbf{x}$ - plane. The functions f, used for representing Γ, are defined by

$$f(\theta) = \hat{f}(\theta)/L,\ 0 \leq \theta \leq 2\pi,$$

in the case of the cylinder problem and by

$$f(x_1^*) = \hat{f}(x_1)/L,\ 0 \leq x_1^* (= x_1/L) \leq 1,$$

in the case of the other problems, where $\hat{f}$ represents the "real" free surface (in the $\mathbf{x}$ - plane).)

If the velocity is sufficiently small so that the nonlinear term in (2.8) may be ignored, we obtain the (dimensionless) Stokes equations :

(2.10) $-S_{ij,j} = f_i,\ i = 1, 2,$

(2.11) $u_{i,i} = 0,$

where $S_{ij}(\mathbf{u}, p)$ is defined as in (2.8).

2.3 Boundary conditions

The following boundary conditions (given in dimensionless form) apply to problems I - IV.

(2.12) $\mathbf{u} = \mathbf{u}_0$ on Σ (prescribed velocity).

Here $\mathbf{u}_0 = (u_{01}, u_{02})$ is a given vector function representing the flow across Σ. Apart from regularity requirements (given in the next chapter), $\mathbf{u}_0$ must satisfy the following conditions (listed with the numbers of the relevant problems):

I.	$\mathbf{u}_0(\mathbf{x}) = (-x_2, x_1)$, $\mathbf{x} \in \Sigma$ (velocity of rotating cylinder ; thus (2.12) is simply a no-slip condition);
II.	$\mathbf{u}_0(x_1 + 1, x_2) = \mathbf{u}_0(\mathbf{x})$, $\mathbf{x} \in \Sigma'$ (periodicity in x_1);
II, IV.	$\int_\Sigma \mathbf{u}_0 \cdot \mathbf{n}\, ds = 0$ (zero total effect of sinks and sources);
III, IV.	$\mathbf{u}_0$ must be compatible with the boundary conditions applied at the sections of $\partial\Omega$ adjoining Σ.

For the container problem it is usually assumed that

(2.13a) $\mathbf{u} = \mathbf{0}$ on Λ (no-slip condition).

However, it will be shown that well-posed boundary value problems can be formulated by using the boundary condition

$$(2.13b) \quad \begin{cases} u_i n_i = 0 & \text{on } \Lambda \text{ (slip condition)} \\ S_{ij} n_j t_i = 0 & \text{on } \Lambda \text{ (tangential stress condition)} \end{cases}$$

or a combination of (2.13a) and (2.13b) applied to separate portions of Λ. Here, and elsewhere, $\mathbf{n} = (n_1, n_2)$ and $\mathbf{t} = (t_1, t_2)$ denote the outward unit normal and tangential vectors to $\partial\Omega$, respectively. The orientation of $\mathbf{t}$ is taken to be clockwise.

In the case of the die-swell problem, the condition of symmetry with respect to the x_1 - axis is expressed by (2.13b) on Λ_s. At Λ_w it is again possible to choose between (2.13a) and (2.13b). (Note that whenever (2.13a) is applied, it is natural to incorporate it into (2.12) by relabelling the relevant portion of Λ as part of Σ and setting $\mathbf{u}_0 = \mathbf{0}$ there.) At Λ_o it is supposed that the flow is parallel to the x_1 - axis and that there is no diffusive outflow of momentum:

$$\text{(2.13c)} \quad \begin{cases} (u_2 =) u_i t_i = 0 & \text{on } \Lambda_o \text{ (kinematic condition)} \\ S_{ij} n_j n_i = 0 & \text{on } \Lambda_o \text{ (normal stress condition).} \end{cases}$$

Observe that the use of boundary conditions on Λ_0 and Λ_1 in the channel problem is avoided by requiring $\mathbf{u}$ to be periodic (in x_1) in Ω'.

In stationary situations the free boundary is a streamline:

$$\text{(2.14)} \quad u_i n_i = 0 \quad \text{on} \quad \Gamma \text{ (kinematic condition).}$$

Moreover, a balance of forces must be fulfilled on the free boundary :

$$\text{(2.15)} \quad S_{ij} n_j t_i = 0 \quad \text{on} \quad \Gamma \text{ (tangential stress condition),}$$

$$\text{(2.16)} \quad \tfrac{1}{R} = (Re.Oh)^2 (S_{ij} n_j n_i + p_a) \text{ (normal stress condition) on } \Gamma.$$

Here $\frac{1}{R}$ is the curvature operator, p_a is the pressure of the surrounding atmosphere (or inviscid fluid) and $Oh = \mu/\sqrt{\rho\sigma L}$ is the Ohnesorge number, with σ the coefficient of surface tension. The curvature operator assumes the form

$$\frac{1}{R} = \frac{f^2 + 2(f')^2 - ff''}{(f^2 + (f')^2)^{\frac{3}{2}}} \quad \left(' \equiv \frac{d}{d\theta} \right)$$

in the cylinder problem, and in the other problems it becomes

$$\frac{1}{R} = \left[\frac{f'}{(1+(f')^2)^{\frac{1}{2}}}\right]' = \frac{f''}{(1+(f')^2)^{\frac{3}{2}}} \quad \left(' \equiv \frac{d}{dx_1}\right).$$

Since the position of the free boundary Γ (equivalently, f) is determined by the solution of a second-order ordinary differential equation (see (2.16)), two boundary conditions for the position of Γ are necessary. For problems I – IV these conditions are :

(2.17) I. $f(\theta + 2\pi) = f(\theta)\ \forall\, \theta \in \Re$ (hence, $f(0) = f(2\pi)$ and $f'(0) = f'(2\pi)$);

II. $f(x+1) = f(x)\ \forall\, x \in \Re$ (periodicity condition);

III. $f(0) = b$ (fixed seperation point), $f'(1) = 0$ (vanishing slope at outflow);

IV. $f'(0) = -c$, $f'(1) = c$ (contact angle condition).

In the case of problems I, II and IV, f is determined only up to an additive constant by (2.16) and (2.17). To fix the position of Γ uniquely, the volume of the liquid in the container is prescribed:

(2.18) $meas(\Omega) = \int_\Omega dx_1 dx_2 = Vol$ (volume constraint).

We can now formulate (for each of the settings I – IV) the free boundary problem:

(FBP) Determine Γ (or f), u and p such that equations (2.8) and (2.9) (or (2.10) and (2.11)) hold in Ω and the boundary conditions (2.12) - (2.18) are satisfied.

Note that the number of boundary conditions at the free boundary (see (2.14) - (2.16)) is equal to three instead of two, which would be the case for a fixed domain problem. The extra boundary condition is necessary since the position of the free boundary is an additional unknown. It is precisely this property of (FBP) on which the so-called splitting method hinges:

(a) By treating the position of Γ as a given (but arbitrary) entity and ignoring one of the boundary conditions at Γ, namely (2.16), one obtains a well-posed (uniquely solvable) boundary value problem in $\mathbf{u}$ and p. This establishes a mapping of the form $\Gamma \rightarrow (\mathbf{u}, p)$.

(b) Using the properties of this map, one proves that (2.16) has a unique solution. Calculation of the corresponding pair $(\mathbf{u}, p)$ then solves (FBP) (cf. Pukhnachov (1972a, 1972b, 1975), Socolescu (1978a, 1978b), Solonnikov (1980), Jean (1980)).

Our aim is to investigate the degree to which this strategy can be pursued successfully by using purely variational (weak) methods. We begin by studying step (a). For every fixed Γ (or f) satisfying the regularity requirements and conditions (2.17) - (2.18), we set the corresponding auxiliary problem:

(Aux) Determine $\mathbf{u}$ and p such that the equations (2.8) and (2.9) (alternatively, (2.10) and (2.11)) hold in Ω and boundary conditions (2.12) – (2.15) are satisfied.

In the next chapter the function spaces from which $\mathbf{u}$ and p are to be found, and the sense in which the equations and boundary conditions of (Aux) are meant to be satisfied, will be made precise.

3 Variational Form of the Fixed Domain Problem

In order to establish existence results for problem (Aux) it is necessary to specify in an exact manner the degree of smoothness required from the data and (possible) solutions. This will be done by introducing suitable classes of admissible functions and deriving properties of the corresponding classes of boundary values (section 3.1). Then a general Green's formula will be derived (section 3.2) by means of which a variational (or weak) form of (Aux) is obtained (section 3.3). The approach followed in this chapter is essentially the same as that of Kikuchi and Oden (1988), pp. 83–93.

3.1 Trace theorems

Let Ω be a bounded open subset of $\Re^2$ with a Lipschitz continuous boundary $\partial\Omega$. The space $L^2(\Omega)$ of (equivalence classes of) Lebesgue - square integrable functions is a Hilbert space with inner product and norm defined by

$$(3.1)\quad (u,\ v)_{0,\Omega} = \int_\Omega uv\,dx, \quad \|u\|_{0,\Omega} = \sqrt{(u,u)_{0,\Omega}}.$$

For integer $m \geq 0$ the Sobolev space defined by

$$(3.2)\quad H^m(\Omega) = \{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega)\ \forall \mid \alpha \mid \leq m\}$$

is a Hilbert space with inner product and norm defined by

$$(3.3)\quad (u,v)_{m,\Omega} = \sum_{|\alpha| \leq m} (D^\alpha u,\ D^\alpha v)_{0,\Omega}, \quad \|u\|_{m,\Omega} = \sqrt{(u,u)_{m,\Omega}}.$$

Here $\alpha = (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$ for integers $\alpha_1, \alpha_2 \geq 0$. $D^\alpha u$ denotes the distributional derivative

$$\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha}\partial x_2^{\alpha}}.$$

(We shall not give a comprehensive treatment of the Sobolev spaces here. Details can be found in references like Lions and Magenes (1972), Adams (1975) or Grisvard (1985). The basic idea is the following:

Let $C_0^\infty(\Omega)$ denote the linear space of infinitely differentiable functions defined on Ω with compact support in Ω. The space $\mathcal{D}(\Omega)$ of test functions is defined as $C_0^\infty(\Omega)$ equipped with the standard locally convex topology.

The space $\mathcal{D}'(\Omega)$ of distributions on Ω is defined as the topological dual of $\mathcal{D}(\Omega)$ and is provided with the strong dual topology (cf. Lions and Magenes (1972), p.2, and Oden and Carey (1983), p.9).

Let $< \cdot, \cdot >$ denote the duality paining between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$. If $u \in \mathcal{D}'(\Omega)$, its derivative $\mathcal{D}^\alpha u$ is defined by

$$< \mathcal{D}^\alpha u, \phi > = (-1)^{|\alpha|}. < u, \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha}\partial x_2^{\alpha}} > \quad \forall\, \phi \in \mathcal{D}(\Omega).$$

If $u \in \mathcal{D}'(\Omega)$ and there exists a function $\hat{u} \in L^2(\Omega)$ such that $< u, \phi > = \int_\Omega \hat{u}\phi\, dx \quad \forall\, \phi \in \mathcal{D}(\Omega)$, then we identify u with $\hat{u}$ and write $u \in L^2(\Omega)$. The definition of $H^m(\Omega)$ and its inner product is to be understood in this sense.)

We shall mainly be interested in the space of vector-valued functions $\mathbf{v}$ with components v_i in $H^1(\Omega)$:

$$\begin{aligned} H^1(\Omega)^2 &= \{\mathbf{v} = (v_1, v_2) \mid v_i \in H^1(\Omega),\ i = 1,2\} \\ &= \{\mathbf{v} = (v_i, v_2) \mid v_i \in L^2(\Omega),\ v_{i,j} \in L^2(\Omega),\ i,j = 1,2\}, \\ (\mathbf{u}, \mathbf{v})_{1,\Omega} &= (u_i, v_i)_{1,\Omega} = (u_i, v_i)_{0,\Omega} + (u_{i,j}, v_{i,j})_{0,\Omega}, \text{ etc.} \end{aligned}$$

For the treatment of boundary conditions a precise meaning must be assigned to the "boundary values" of the functions v_i in $H^1(\Omega)$. We begin with the following definition:

(3.4) Let I be an open subset of $\Re$. Then we denote by $H^{\frac{1}{2}}(I)$ the space of all distributions u defined in I such that :

$u \in L^2(I)$ and $\int_g \int_g \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy < +\infty$.

Since $\partial\Omega$ is assumed to be Lipschitz continuous, it follows from definition (2.1) that we can view $\partial\Omega$ locally as a 1-dimensional submanifold of $\Re^2$ by means of the mapping $\Phi(y_1) = (y_1, \phi(y_1))$ from O' onto $\partial\Omega \cap O$. We set the following definition :

(3.5) A distribution u on $\partial\Omega$ belongs to $L^2(\partial\Omega)$ (respectively $H^{\frac{1}{2}}(\partial\Omega)$) if $u \cdot \Phi$ belongs to $L^2(O' \cap \Phi^{-1}(\partial\Omega \cap O))$ (respectively $H^{\frac{1}{2}}(O \cap \Phi^{-1}(\partial\Omega \cap O))$, as defined in (3.4)) for all possible O and ϕ fulfilling the assumptions in definition (2.1).

Since Ω is bounded (so that there exists an atlas $\{(O_i,\ \Phi_i)\}_{i=1}^m$ of $\partial\Omega$ such that each pair $(O_i,\ \Phi_i)$ satisfies the hypotheses of definition (2.1)), $L^2(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ are Hilbert spaces with inner products and norms defined by

(3.6) $(u,\ v)_{0,\partial\Omega} = \int_{\partial\Omega} uv\, ds, \quad \|u\|_{0,\partial\Omega} = \sqrt{(u,u)_{0,\partial\Omega}}$,

(3.7) $(u,\ v)_{\frac{1}{2},\partial\Omega} = (u,\ v)_{0,\partial\Omega} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{[u(x)-u(y)][v(x)-v(y)]}{\|x-y\|^2}\, ds(x)\, ds(y)$,

$\|u\|_{\frac{1}{2},\partial\Omega} = \sqrt{(u,u)_{\frac{1}{2},\partial\Omega}}$,

respectively, where ds denotes the arc length along $\partial\Omega$. □

The importance of $H^{\frac{1}{2}}(\partial\Omega)$ is due to the classical trace theorem for $H^1(\Omega)$:

Proposition 3.1 Let Ω be a bounded open subset of $\Re^2$ with a Lipschitz continuous boundary $\partial\Omega$. The operator γ defined by

(3.8) $\gamma(v) = v\,|_{\partial\Omega}, \quad v \in C^1(\bar{\Omega})$, has a unique extension $\gamma \in \mathcal{L}(H^1(\Omega), H^{\frac{1}{2}}(\partial\Omega))$, which is surjective. This operator has a continuous linear right inverse.

Proof See, e.g., Lions and Magenes (1972), p. 39. $\square$

In (3.5) - (3.8), $\partial\Omega$ may be replaced by any open subset Γ of $\partial\Omega$. We shall denote the corresponding trace operator by γ_Γ.

Let Ω be as above. Suppose that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma}$ where Γ and Σ are non-empty disjoint open subsets of $\partial\Omega$. The spaces defined in (3.5) are sometimes inappropriate for dealing with problems with mixed boundary conditions (e.g., problems III and IV). For instance, if $u \in H^{\frac{1}{2}}(\Gamma)$ and $\tilde{u}$ is the extension by zero of u to Σ, then in general $\tilde{u}$ is not the trace of a function in $H^1(\Omega)$, i.e., $\tilde{u} \notin H^{\frac{1}{2}}(\partial\Omega)$. This difficulty is overcome by restricting u to a special subspace of $H^{\frac{1}{2}}(\Gamma)$ which is defined as follows (cf. Lions and Magenes (1972), pp. 57, 66) :

(3.9) Let a function of ρ be defined on $\bar{\Gamma}$ such that ρ is sufficiently smooth, positive on Γ and vanishes on the boundary $\partial\Gamma$ of Γ at a rate

$$r = \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\rho(\mathbf{x})}{dist(\mathbf{x},\, \partial\Gamma)} \neq 0,\ \mathbf{x}_0 \in \partial\Gamma.$$

Then $H^{\frac{1}{2}}_{00}(\Gamma) = \left\{u \in H^{\frac{1}{2}}(\Gamma) \mid \rho^{-\frac{1}{2}}.u \in L^2(\Gamma)\right\}$ is a Hilbert space with inner product and norm defined by

$$(u,\, v)_{00,\Gamma} = (u,\, v)_{\frac{1}{2},\Gamma} + (\rho^{-\frac{1}{2}}u, \rho^{-\frac{1}{2}}v)_{0,\Gamma}, \quad \|u\|_{00,\Gamma} = \sqrt{(u,u)_{00,\Gamma}}. \square$$

Let U denote the closed subspace of $H^1(\Omega)$ given by $U = \{u \in H^1(\Omega) \mid \gamma_\Sigma(u) = 0\}$. Let γ^0_Γ be the restriction of the trace operator $\gamma_\Gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ to U. Then we have the following trace theorem :

Proposition 3.2 Let $\gamma^0_\Gamma : U \to H^{\frac{1}{2}}(\Gamma)$ be the operator defined above. Then γ^0_Γ maps U onto $H^{\frac{1}{2}}_{00}(\Gamma)$ and $\gamma^0_\Gamma \in \mathcal{L}(U, H^{\frac{1}{2}}_{00}(\Gamma))$. This map has a continuous linear right inverse.

Proof It is not difficult to show that the norm $\|u\|_{00,\Gamma}$ on $H_{00}^{\frac{1}{2}}(\Gamma)$ is equivalent to $\|\tilde{u}\|_{\frac{1}{2},\partial\Omega}$, where $\tilde{u}$ denotes the extension by zero of u to $\partial\Omega$. In fact, the natural definition of $H_{00}^{\frac{1}{2}}(\Gamma)$ and its norm is the one given in Grisvard (1985), pp. 18–19 :

$$H_{00}^{\frac{1}{2}}(\Gamma) = \left\{u \in H^{\frac{1}{2}}(\Gamma) \mid \tilde{u} \in H^{\frac{1}{2}}(\partial\Omega)\right\}, \; \|u\|_{00,\Gamma} = \|\tilde{u}\|_{\frac{1}{2},\partial\Omega}.$$

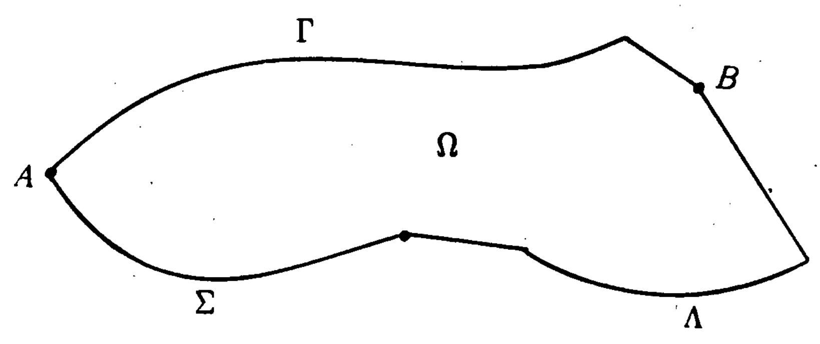
This corresponds to the choice

$$\rho(\mathbf{x})^{-1} = 2\int_{\partial\Omega\setminus\Gamma} \frac{ds(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}$$

in (3.9).

This implies that γ_Γ^0 maps U into and onto $H_{00}^{\frac{1}{2}}(\Gamma)$ (cf. Corollary 1.4.4.10 in Grisvard (1985)). The remaining statements follow immediately from Proposition 3.1. □

Suppose that $\partial\Omega = \bar{\Gamma} \cup \bar{\Sigma} \cup \bar{\Lambda}$ with Γ, Σ and Λ non-empty, disjoint , open and connected, $\bar{\Gamma} \cap \bar{\Sigma} = \{A\}$ and $\bar{\Gamma} \cap \bar{\Lambda} = \{B\}$. Let $U' = \{u \in H^1(\Omega) \mid \gamma_\Sigma(u) = 0\}$.



In this situation the spaces $H^{\frac{1}{2}}(\Gamma)$ and $H_{00}^{\frac{1}{2}}(\Gamma)$ are both inappropriate for characterizing the traces on Γ of functions in U'. We need a space "intermediate" to them :

(3.10) Let ρ_A be a function defined on $\bar{\Gamma}$ with properties identical to those of the function ρ in (3.9), but with $\partial\Gamma$ replaced by $\{A\}$ and ρ_A approaching, say, 1 at B.

Then $H_A^{\frac{1}{2}}(\Gamma) = \left\{u \in H^{\frac{1}{2}}(\Gamma) \mid \rho_A^{-\frac{1}{2}}.u \in L^2(\Gamma)\right\}$ is a Hilbert space with inner product and norm defined as in (3.9) with ρ replaced by ρ_A.

It follows from arguments identical to those in the proof of Proposition 3.2 that γ_Γ maps U' onto $H_A^{\frac{1}{2}}(\Gamma)$ and $\gamma_\Gamma^A \in \mathcal{L}(U', H_A^{\frac{1}{2}}(\Gamma))$ where γ_Γ^A is the restriction of γ_Γ to U'.

Definitions and results analogous to these apply to vector-valued functions $\mathbf{v}$ in $H^1(\Omega)^2$ and their boundary values. In order to deal with the boundary conditions of problems I - IV it is necessary to decompose the traces $\gamma(\mathbf{v})$ in $H^{\frac{1}{2}}(\partial\Omega)^2$ of such functions into well-defined normal and tangential components. The following theorem shows that this can be done when the boundary $\partial\Omega$ is sufficiently smooth.

Proposition 3.3 Let Ω be a bounded open subset of $\Re^2$ with a Lipschitz continuous boundary $\partial\Omega$. Let Γ be an open subset of $\partial\Omega$ of class $C^{1,\alpha}$, $\alpha > \frac{1}{2}$.

(a) Then every $\mathbf{w} \in H^{\frac{1}{2}}(\Gamma)^2$ has a unique decomposition $\mathbf{w} = w_n\mathbf{n} + w_t\mathbf{t}$ with $w_n,\ w_t \in H^{\frac{1}{2}}(\Gamma)$. The map $(w_1,\ w_2) \to (w_n,\ w_t)$ is an isomorphism form $H^{\frac{1}{2}}(\Gamma)^2$ onto itself.

(b) There exist uniquely determined operators $\gamma_{\Gamma n},\ \gamma_{\Gamma t} \in \mathcal{L}(H^1(\Omega)^2, H^{\frac{1}{2}}(\Gamma))$ such that
$\gamma_\Gamma(\mathbf{v}) = \gamma_{\Gamma n}(\mathbf{v})\mathbf{n} + \gamma_{\Gamma t}(\mathbf{v})\mathbf{t}\ \ \forall\, \mathbf{v} \in H^1(\Omega)^2$ and
$\gamma_{\Gamma n}(\mathbf{v}) = \mathbf{v}|_\Gamma \cdot \mathbf{n},\ \gamma_{\Gamma t}(\mathbf{v}) = \mathbf{v}|_\Gamma \cdot \mathbf{t}\ \ \forall\, \mathbf{v} \in C^1(\bar{\Omega})^2$.

Furthermore, the map $(\gamma_{\Gamma n},\ \gamma_{\Gamma t}) : H^1(\Omega)^2 \to H^{\frac{1}{2}}(\Gamma)^2$ is surjective and has a continuous linear right inverse.

(c) Results analogous to those in (a) and (b) apply when $H^1(\Omega)$, $H^{\frac{1}{2}}(\Gamma)$ and γ_Γ are replaces by U, $H_{00}^{\frac{1}{2}}(\Gamma)$ and γ_Γ^0, or U', $H_A^{\frac{1}{2}}(\Gamma)$ and γ_Γ^A.

Proof (a) By using the system of local charts of Γ it can be shown that the outward unit vector $\mathbf{n} = (n_1,\ n_2)$ exists everywhere on Γ and $n_i \in C^{0,\alpha}(\bar{\Gamma})$, $i = 1,2$. Since $\alpha > \frac{1}{2}$, it follows (cf. Theorem 1.4.1.1 in Grisvard (1985)) that , for every $v \in H^{\frac{1}{2}}(\Gamma)$, $vn_i \in H^{\frac{1}{2}}(\Gamma)$ and there exists a constant $K = K(\mathbf{n})$ such that $\|vn_i\|_{\frac{1}{2},\Gamma} \le K.\|v\|_{\frac{1}{2},\Gamma}$, $i = 1,2$.

Thus, for $\mathbf{w} \in H^{\frac{1}{2}}(\Gamma)^2$, $w_n = w_i n_i \in H^{\frac{1}{2}}(\Gamma)$ and $w_t = w_i t_i \in H^{\frac{1}{2}}(\Gamma)$ where $\mathbf{t} = (n_2, -n_1)$. It is easy to show that $\mathbf{w} = w_n \mathbf{n} + w_t \mathbf{t}$ and that this decomposition is unique. Moreover, the map $\mathbf{w} \longrightarrow (w_n, w_t)$ is linear, continuous (by the inequalities above), injective and surjective (by the same argument as above).

(b) This follows directly from Proposition 3.1 and (a).

(c) If $v \in H^{\frac{1}{2}}_{00}(\Gamma)$ then $v \in H^{\frac{1}{2}}(\Gamma)$ and thus $vn_i \in H^{\frac{1}{2}}(\Gamma)$, $i = 1,2$, as in the proof of (a). Since $\int_\Gamma (vn_i)^2/\rho \, ds \leq \int_\Gamma (v.1)^2/\rho \, ds < \infty$, $\|vn_i\|_{00,\Gamma} < \infty$ and thus $vn_i \in H^{\frac{1}{2}}_{00}(\Gamma)$, $i = 1,2$. Using this fact, the analogue of (a) is proved as above. The analogue of (b) then follows from Proposition 3.2. The proof is identical for $H^{\frac{1}{2}}_{A}(\Gamma)$. □

For problems I – IV we shall henceforth assume that Γ and the other portions of $\partial\Omega$ on which Neumann boundary conditions are applied are (separately) of class $C^{1,1}$.

The space of admissible flows is defined as the subspace of $H^1(\Omega)^2$ consisting of the velocity fields $\mathbf{v}$ satisfying the Dirichlet boundary conditions specified at Λ and Σ (with $\mathbf{u}_0$ replaced by $\mathbf{0}$) for $\mathbf{u}$ in (Aux). Hence, for the respective problems, we define:

$$
\begin{aligned}
(3.11) \quad & \text{I.} \quad V_1 = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\}; \\
& \text{II.} \quad V_2 = \Big\{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v} = \mathbf{v}'|_\Omega,\ \mathbf{v}' \in H^1(\Phi' \cap \Omega')^2, \\
& \qquad \mathbf{v}'(x_1+1, x_2) = \mathbf{v}'(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Omega,\ \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\Big\} \\
& \qquad \text{where } \Phi' = \{\mathbf{x} \in \Re^2 \mid 0 < x_1 < 2\}; \\
& \text{III.} \quad V_3 = \Big\{\mathbf{v} \in H'(\Omega)^2 \mid \gamma_{\Sigma \cup \Lambda_w}(\mathbf{v}) = \mathbf{0},\ \gamma_{\Lambda_s n}(\mathbf{v}) = 0,\ \gamma_{\Lambda_o t}(\mathbf{v}) = 0\Big\}; \\
& \text{IV.} \quad V_4 = \Big\{\mathbf{v} \in H'(\Omega)^2 \mid \gamma_{\Sigma \cup \Lambda}(\mathbf{v}) = \mathbf{0}\Big\}.
\end{aligned}
$$

In the case of a slip condition at Λ_w, the first constraint in V_3 is replaced by $\gamma_\Sigma(\mathbf{v}) = \mathbf{0}$, $\gamma_{\Lambda_w n}(\mathbf{v}) = 0$. The same idea applies to V_4. The periodicity condition on $\mathbf{v}$ in V_2 $\Big($i.e. $v'_{i,j} \in L^2(\Phi' \cap \Omega')$, $i,j = 1,2\Big)$ is equivalent to the "intrinsic" condition $\gamma_{\Lambda_0}(\mathbf{v})(x_2) = \gamma_{\Lambda_1}(\mathbf{v})(x_2)$ for a.e. x_2. (This follows easily from the definition of the distribution $v'_{i,j} \in \mathcal{D}(\Phi' \cap \Omega')$.)

For $i = 1, ..., 4, V_i$ is a closed subspace of $H^1(\Omega)^2$ and therefore a Hilbert space with the norm $\|\cdot\|_{1,\Omega}$. (Let γ_0 denote the trace map from $H^1(\Omega)^2$ onto $H^{\frac{1}{2}}(\partial\Omega)^2$. Then there exists an operator $T_i \in \mathcal{L}(H^{\frac{1}{2}}(\partial\Omega)^2, X$, where X is a space of (restrictions of) traces, such that $V_i = ker(T_i \cdot \gamma_0)$, as can easily be seen from the definition of V_i. Thus V_i is the kernel of a bounded linear operator defined on $H^1(\Omega)^2$. (For V_2 we replace $H^1(\Omega)^2$ and $H^{\frac{1}{2}}(\partial\Omega)^2$ by the corresponding subspaces of 1− periodic elements, which are closed, in this argument.))

Since $meas(\Sigma) > 0$, an equivalent norm on V_i is given by $| \mathbf{v} |_1 = \sqrt{(v_{k,j}, v_{k,j})_{0,\Omega}}$ (cf. Lemma 3.1 in Girault and Raviart (1986)).

Moreover, for $i = 1, ..., 4$, the set $\gamma_0(V_i)$ of traces of admissible flows is a closed subspace of $H^{\frac{1}{2}}(\partial\Omega)^2$ and therefore a Hilbert space. (Let $\mathbf{w} = \gamma_0(\mathbf{v})$. Then $\mathbf{v} \in V_i = ker(T_i \cdot \gamma_0)$ iff $\mathbf{w} \in ker\ T_i$. Since γ_0 is surjective, $\gamma_0(V_i) = ker\ T_i$.)

We shall now define, for $i = 1, ..., 4$, a Hilbert space Z_i, which is isomorphic to $\gamma_0(V_i)$ and represents the nonzero (or unprescribed) parts of the traces in $\gamma_0(V_i)$, and a corresponding surjective trace operator $\gamma_i \in \mathcal{L}(V_i, Z_i)$:

(3.12) I. $Z_1 = H^{\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, $\gamma_1 = (\gamma_{\Gamma n}, \gamma_{\Gamma t})$;

II. $Z_2 = H_1^{\frac{1}{2}}(\Gamma) \times H_1^{\frac{1}{2}}(\Gamma)$, $\gamma_2 = (\gamma^1_{\Gamma n}, \gamma^1_{\Gamma t})$

where $H_1^{\frac{1}{2}}(\Gamma) = \left\{ v \in H^{\frac{1}{2}}(\Gamma) \mid v = v'|_\Gamma,\ v' \in H^{\frac{1}{2}}(\Phi' \cap \Gamma'), v'(x_1 + 1, x_2) = v'(\mathbf{x}) \text{ for a.e. } \mathbf{x} \in \Gamma \right\}$ and γ^1 denotes the restriction of γ to V_2;

III. $Z_3 = H^{\frac{1}{2}}_{00}(\Gamma) \times Y_3$, $\gamma_3 = \left(\gamma^0_{\Gamma n}, \gamma^A_{\Gamma t}, \gamma_{\Lambda_o n}, \gamma^D_{\Lambda_s t}\right)$

where Y_3 is the subspace of $H^{\frac{1}{2}}_A(\Gamma) \times H^{\frac{1}{2}}(\Lambda_o) \times H^{\frac{1}{2}}_D(\Lambda_s)$ consisting of elements(u, v, w) such that $\mathbf{a} \in H^{\frac{1}{2}}_{00}(\Gamma')^2$, $\Gamma' = \Gamma \cup \Lambda_o \cup \Lambda_s$, if $\mathbf{a}$ is defined by

$$\mathbf{a} = \begin{cases} ut \text{ on } \Gamma \\ vn \text{ on } \Lambda_o \\ wt \text{ on } \Lambda_s; \end{cases}$$

IV. $Z_4 = H^{\frac{1}{2}}_{00}(\Gamma) \times H^{\frac{1}{2}}_{00}(\Gamma)$, $\gamma_4 = (\gamma^0_{\Gamma n}, \gamma^0_{\Gamma t})$.

The condition $\|\mathbf{a}\|_{00,\Gamma'} < +\infty$ in (3.12 III) is equivalent to $\|\mathbf{a}\|_{\frac{1}{2},\Gamma'} < +\infty$ and can be expressed as compatibility conditions to th effect that "$\mathbf{u}\mathbf{t} = \mathbf{v}\mathbf{n}$ at B" and "$\mathbf{v}\mathbf{n} = \mathbf{w}\mathbf{t}$ at C". Similarly, the periodicity condition $\|v'\|_{\frac{1}{2},\Phi' \cap \Gamma'} < +\infty$ in (3.12 II) is equivalent to a condition of the form "$v(0) = v(1)$". The precise formulation can be found in Theorem 1.5.2.3 (c) of Grisvard (1985). One consequence of this is that Y_3 cannot be expressed in the form $H_1 \times H_2 \times H_3$, the significance of which will become clear in the next section. Note that in each case we have been able to isolate the space corresponding to γ_{Γ_n}. However, if a slip condition is applied to Λ_w in problem III, or Λ_0 and/or Λ_1 in problem IV, this is not possible due to the appearance of additional compatibility conditions.

It is easily verified that the spaces Z_i and operators γ_i do have the stated properties; eg., the completeness of Z_3 follows from that of $\gamma_0(V_3)$.

3.2 Green's formulas

In this section a general Green's formula is derived from which the well-known Green's formula involving the Laplace operator and Sobolev spaces is obtained. This result has been established by numerous authors (cf., eg., Baiocchi and Capelo (1984) or Grisvard (1985)) but we shall follow a more general approach, as given in Kikuchi and Oden (1988), which permits a less technical proof.

Let $\mathcal{D}, V, H, Z$ and $\mathcal{S}$ be linear spaces such that

(3.13) V, H, Z and $\mathcal{S}$ are Hilbert spaces with topological duals V', H', Z' and $\mathcal{S}'$,
V is contained in H with a finer topology,
H is a pivot space, i.e., H is identified with its dual,
$\mathcal{D}$ is a topological space contained in V and dense in H.

Let V_0 denote the closure of $\mathcal{D}$ in V. Then we have the inclusions

$$\mathcal{D} \overset{d}{\subset} V_0 \subset V \overset{d}{\subset} H \equiv H' \subset V' \subset V_0'$$

where $\overset{d}{\subset}$ denotes the inclusion is dense.

Let A be a continuous linear operator from V into $\mathcal{S}$ and let A_0 be its restriction to V_0 :

(3.14) $A \in \mathcal{L}(V, \mathcal{S}),\ A_0 \in \mathcal{L}(V_0, \mathcal{S})$ and $A_0 v = Av\ \ \forall v \in V_0.$

Denote the duality pairings on $\mathcal{S}' \times \mathcal{S}$, $V' \times V$ and $H' \times H$ by $[\cdot,\cdot]$, $<\cdot,\cdot>$ and $(\cdot,\cdot)$, respectively. Then the *transpose* of A is the operator $A^* \in \mathcal{L}(\mathcal{S}', V')$ defined by

(3.15) $< A^*\tau, v > = [\tau, Av]\ \ \forall\, \tau \in \mathcal{S}'$ and $\forall\, v \in V.$

Similarly, the transpose $A_0^* \in \mathcal{L}(\mathcal{S}', V_0')$ of A_0 is defined by

(3.16) $< A_0^*\tau, v > = [\tau, A_0 v]\ \ \forall\, \tau \in \mathcal{S}'$ and $\ \ \forall\, v \in V_0.$

Finally, we define the space $\mathcal{T}$ and its inner product by

(3.17) $\mathcal{T} = \left\{\tau \in \mathcal{S}' \mid A_0^*\tau \in H'\right\}$ and

(3.18) $((\tau,\sigma)) = (\tau,\sigma)_{\mathcal{S}'} + (A_0^*\tau, A_0^*\sigma)_{H'}.$

Here $(\cdot,\cdot)_{\mathcal{S}'}$ and $(\cdot,\cdot)_{H'}$ are the inner products on $\mathcal{S}'$ and H' induced by the Riesz isomorphisms $j_S : \mathcal{S} \rightarrow \mathcal{S}'$ and $J_H : H \rightarrow H'$, respectively.

<u>Lemma 3.1</u> $\mathcal{T}$ is a Hilbert space with the inner product $((\cdot,\cdot))$ and A_0^* is a well-defined continuous linear map from $\mathcal{T}$ into H'.

<u>Proof</u> By definition, $\tau \in \mathcal{T}$ iff $A_0^*\tau \in V_0'$ can be extended as an element of H', i.e., iff there exists a $q \in H'$ such that $< A_0^*\tau, v > = (q, v)\ \ \forall\, v \in V_0$. Suppose that $q,\ q' \in H'$ with $(q,v) = (q',v)\ \ \forall\, v \in V_0$. Since V_0 is dense in H, there exists a sequence (v_n) in

V_0 which converges to $j_H^{-1}(q - q')$ in H. A simple argument shows that this implies that $q = q'$. Hence, for every $\tau \in \mathcal{T}, A_0^* \tau \in H'$ is uniquely determined; i.e., $A_0^* \in L(\mathcal{T}, H')$ is well-defined.

It follows easily from the linearity of A_0^* and the fact that $\mathcal{S}'$ and H' are inner product spaces with $(\cdot,\cdot)_{\mathcal{S}'}$ and $(\cdot,\cdot)_{H'}$ that $\mathcal{T}$ is a linear space and that $((\cdot,\cdot))$ is an inner product on it.

$A_0^* \in \mathcal{L}(\mathcal{T}, H')$ because $\|A_0^*\tau\|_{H'}^2 \leq \|\tau\|_{\mathcal{S}'}^2 + \|A_0^*\tau\|_{H'}^2 = \|\tau\|_{\mathcal{T}}^2 \ \forall \tau \in \mathcal{T}$. It remains to show that $\mathcal{T}$ is complete.

The graphs of A_0^* and $-A_0$ are defined by

$$G(A_0^*) = \left\{(\tau, h) \in \mathcal{S}' \times H' \mid A_0^*\tau = h\right\} \text{ and}$$

$$G(A_0) = \{(s, v) \in \mathcal{S} \times H \mid -A_0 v = s\}, \text{ respectively.}$$

Note that τ is restricted to $\mathcal{T}$ and v to V_0 in these definitions. Duality pairing on $(\mathcal{S} \times H)' \times (\mathcal{S} \times H) = (\mathcal{S}' \times H') \times (\mathcal{S} \times H)$ is characterized by the bilinear form $[\tau, s] + (h, v)$. We shall now show that $G(A_0^*) = G(A_0)^a$ where

$$G(A_0)^a = \left\{(\tau, h) \in \mathcal{S}' \times H' \mid [\tau, s] + (h, v) = 0 \ \forall\, (s, v) \in G(A_0)\right\}$$

is the so-called *annihilator* of $G(A_0)$.

Let $(\tau, h) \in G(A_0)^a$. For every $v \in V_0, (-A_0 v, v) \in G(A_0)$ and thus $0 = [\tau, -A_0 v] + (h, v) = - < A_0^*\tau, v > + (h, v)$. This implies that $A_0^*\tau = h$ and so $(\tau, h) \in G(A_0^*)$. Conversely, let $(\tau, h) \in G(A_0^*)$. For every $(s, v) \in G(A_0)$, $v \in V_0$ and $-A_0 v = s$, so that $[\tau, s] + (h, v) = [\tau, -A_0 v] + (A_0^*\tau, v) = - < A_0^*\tau, v > + < A_0^*\tau, v > = 0$. Thus $(\tau, h) \in G(A_0)^a$.

Therefore $G(A_0^*) = G(A_0)^a$, which is easily proved to be a closed subspace of $\mathcal{S}' \times H'$. Hence $G(A_0^*) = \{(\tau, A_0^*\tau) \mid \tau \in \mathcal{T}\}$ is complete with respect to the $\mathcal{S}' \times H'$ -norm, which means that $\mathcal{T}$ is complete under the norm induced by $((\cdot,\cdot))$. □

<u>Lemma 3.2</u> Let $\gamma \in \mathcal{L}(V, Z)$ be a surjective map from a Hilbert space V onto a Hilbert space Z. Denote $ker\gamma$ $(= \{v \in V \mid \gamma v = 0\})$ by V_0. Then there exists a right inverse $\delta \in \mathcal{L}(Z, V)$ of γ (i.e., $\gamma \cdot \delta$ is the identity map form Z onto itself). For every such $\delta, \delta \cdot \gamma$ is a projection (i.e., $\delta \cdot \gamma \in \mathcal{L}(V, V)$ and $(\delta \cdot \gamma)^2 = \delta \cdot \gamma$) with $ker(\delta \cdot \gamma) = V_0$. Moreover, its transpose $\gamma^* \cdot \delta^*$ is a projection of V onto $V_0^a = \left\{g \in V' \mid < g, v > = 0 \;\; \forall\, v \in V_0\right\}$.

<u>Sketch of Proof.</u> The existence of a right inverse δ can be deduced from the fact that, for any $v_0 \in V_0$, the map $\gamma : v_0 + V_0^\perp \to Z$ is a continuous linear bijection, where $V_0^\perp = \{v \in V \mid (u, v)_V = 0 \;\forall\, u \in V_0\}$ is the orthogonal complement of V_0. The continuity of δ follows from the Open Mapping theorem.

The remaining statements can be proved by elementary arguments using the definition of a transpose and the properties of γ, δ, etc.. □

We can now establish the following result, which is the main step in deriving a variational formulation of (Aux). With the notation as before, we have :

<u>Theorem 3.2</u> Suppose that $\gamma \in \mathcal{L}(V, Z)$ is a surjective map from V onto Z such that $V_0 = ker\gamma$. Then there exists a uniquely determined operator $\pi \in \mathcal{L}(\mathcal{T}, Z')$ such that the abstract Green's formula

(3.19) $[\tau, Av] - < A_0^*\tau, v > = \ll \pi\tau, \gamma v \gg$

holds $\forall\, \tau \in \mathcal{T}$ and $\forall\, v \in V$, where $\ll \cdot,\cdot \gg$ denotes the duality pairing on $Z' \times Z$.

<u>Proof</u> (Kikuchi and Oden (1988)). By Lemma 3.2 there exists a right inverse $\delta \in \mathcal{L}(Z, V)$ of γ such that $\gamma^* \cdot \delta^*$ is a projection of V' onto $V_0^a = \left\{g \in V' \mid < g, v > = 0 \;\forall v \in V_0\right\}$.

Let $j \in \mathcal{L}(V, H)$ be the imbedding map and $j^* \in \mathcal{L}(H', V')$ its transpose, i.e.,

$$< j^*h, v > = (h, jv) \quad \forall h \in H' \quad \text{and} \quad \forall v \in V.$$

Then $(A^* - j^* \cdot A_0^*)\tau \in V_0^a$ for every $\tau \in \mathcal{T}$:

Let $\tau \in \mathcal{T}$. Then, for every $v \in V_0$,

$$\begin{aligned}
&< (A^* - j^* \cdot A_0^*)\tau, v > = < A^*\tau, v > - < j^* A_0^* \tau, v > \\
&= [\tau, Av] - (A_0^*\tau, jv) \qquad \text{(by definition of } A^* \text{ and } j^*) \\
&= < A_0^*\tau, v > - < A_0^*\tau, v > \qquad \text{(by definition of } A_0^*; v \in V_0) \\
&= 0.
\end{aligned}$$

Let π denote the operator $\pi = \delta^* \cdot (A^* - j^* \cdot A_0^*)$. Then $\pi \in \mathcal{L}(\mathcal{T}, Z')$ since $A_0^* \in \mathcal{L}(\mathcal{T}, H')$ (Lemma 3.1) and $A^* \in \mathcal{L}(\mathcal{T}, V')$ (because $A^* \in \mathcal{L}(\mathcal{S}', V')$ and $\|\tau\|_{\mathcal{S}'} \leq \|\tau\|_{\mathcal{T}} \; \forall \tau \in \mathcal{T}$).

For every $\tau \in \mathcal{T}$, $(A^* - j^* \cdot A_0^*)\tau = (\gamma^* \cdot \delta^*) \cdot (A^* - j^* \cdot A_0^*)\tau = \gamma^*(\pi\tau)$ since $(A^* - j^* \cdot A_0^*)\tau \in V_0^a$ and $(\gamma^* \cdot \delta^*)g = g \; \forall g \in V_0^a$. Thus, $\forall \tau \in \mathcal{T}$ and $\forall v \in V$,

$$\begin{aligned}
&[\tau, Av] - < A_0^*\tau, v > \qquad \text{(where } < A_0^*\tau, v > \text{ denotes } (A_0^*\tau, jv)) \\
&= < A^*\tau, v > - < j^* A_0^*\tau, v > \qquad = < (A^* - j^* \cdot A_0^*)\tau, v > \\
&= < \gamma^*(\pi\tau), v > = \ll \pi\tau, \gamma v \gg \qquad \text{(by definition of } \gamma^*\text{). Hence, (3.19) is proved.}
\end{aligned}$$

It remains to show that π is unique. Let $\pi' \in \mathcal{L}(\mathcal{T}, Z')$ be an operator satisfying (3.19). Then $\ll (\pi - \pi')\tau, \gamma v \gg = 0 \; \forall v \in V$ and $\forall \tau \in \mathcal{T}$. But γ maps V onto Z. Thus $(\pi - \pi')\tau = 0 \; \forall \tau \in \mathcal{T}$, i.e., $\pi - \pi' = 0$. □

Remark The point of Theorem 3.1 becomes more apparent when (3.19) is compared with (3.15) and (3.16). Although for every $\tau \in \mathcal{S}'$, $A_0^*\tau$ is simply the restriction of $A^*\tau$ to V_0, in general $A_0^*\tau \in H' \subset V'$ differs from $A^*\tau$ on $V_0^\perp$. The Green's formula shows how this difference can be expressed in terms of an arbitrary operator γ and space Z satisfying the requirements of the theorem.

We shall now specialize the Green's formula so as to apply to the flow problems I – IV. For each fixed $1 \leq k \leq 4$, we take

(3.20) $\mathcal{D} = \mathcal{D}(\Omega)^2, \ \ V = V_k, \ \ H = L^2(\Omega)^2, \ \ Z = Z_k$ and

$\mathcal{S} = \{\tau = [\tau_{ij}] \mid \tau_{ij} \in L^2(\Omega), \ \tau_{ij} = \tau_{ji}, \ i,j = 1,2\}.$

Here V_k and Z_k are as defined in section 3.1, $\mathcal{S}$ is the space of symmetric stress fields defined on Ω, and Ω is understood to denote Ω_k, the domain in problem K. For simplicity we shall denote quantities of which the definitions are formally identical in the four problems without the subscript k. The inner product on $\mathcal{S}$ is defined by

(3.21) $(\tau,\sigma)_{\mathcal{S}} \ = \ (\tau_{ij},\sigma_{ij})_{0,\Omega}.$

Then $V_0 \ = \ H_0^1(\Omega)^2 \ = \ \{\mathrm{v} \in H^1(\Omega)^2 \mid \gamma_0(\mathrm{v}) = 0\}$ where $\gamma_0 = \gamma_{\partial\Omega}$.

The operators A and γ are defined by

(3.22) $A\mathrm{v} \ = \ D(\mathrm{v}) \ = \ [\frac{1}{2}(v_{i,j} + v_{j,i})]$ and $\gamma \ = \ \gamma_k$.

It is easily verified that all the requirements of Theorem 3.1 are satisfied. Furthermore, for every $\tau \in \mathcal{S}'$,

(3.23) $A_0^*\tau \ = \ -div\ \tau \ = \ -(\tau_{1j,j},\tau_{2j,j}) \in H^{-1}(\Omega)^2 \ = \ (H_0^1(\Omega)')^2.$

(For every $\tau \ = \ [\tau_{ij}] \in \mathcal{S}'$, $< -div\ \tau,\phi > = - < \tau_{ij,j},\phi_i > =$ $< \tau_{ij},\phi_{i,j} > \ = \ < \tau_{ij},D_{ij}(\phi) > \ = \ [\tau,A_0\phi] \ = \ < A_0^*\tau,\phi > \ \ \forall\phi \in \mathcal{D}$, by definition of the distribution $div\ \tau$, the symmetry of τ and the definitions of A_0 and A_0^*. Thus (3.23) follows since $\mathcal{D}$ is dense in V_0.)

Hence, the space $\mathcal{T}$ and its inner product are given by

(3.24) $\mathcal{T} = \left\{\tau \in \mathcal{S}' \mid \ div\ \tau \in (L^2(\Omega)^2)'\right\}$ and

(3.25) $((\sigma,\tau))_{\mathcal{T}} = (\sigma_{ij},\tau_{ij})_{L^2(\Omega)'} + (\sigma_{ij,j},\tau_{ij,j})_{L^2(\Omega)'}.$

3.3 Weak form of (Aux)

In view of our goal of solving (Aux) under the weakest possible regularity constraints on the data, while posing the problem in terms of the spaces introduced in section 3.1, the natural formulation of (Aux) is the following :

Given a domain Ω satisfying the regularity requirements set out in section 3.1, $\mathbf{f} \in L^2(\Omega)^2$ and $\mathbf{u}_0 \in H^{\frac{1}{2}}(\Sigma)^2$ satisfying the relevant conditions given in section 2.3, find $\mathbf{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$ such that

(3.30) $e.u_j u_{i,j} - \tau_{ij,j} = f_i$ a.e. in Ω, $i = 1,2$, where $\tau_{ij} = S_{ij} = -p\delta_{ij} + \frac{2}{Re}(u_{i,j} + u_{j,i}), e = 1$ for the nonlinear problem and $e = 0$ for the Stokes problem,

(3.31) $u_{i,i} = 0$ a.e. in Ω,

(3.32) $\gamma_\Sigma(\mathbf{u}) = \mathbf{u}_0$,

(3.33) $\gamma_{\Gamma n}(\mathbf{u}) = 0$,

(3.34) $\gamma_\Gamma(\tau_{ij})n_j t_i = 0$,

(3.35) II. $\gamma_{\Lambda_0}(\mathbf{u})(x_2) = \gamma_{\Lambda_1}(\mathbf{u})(x_2)$ for a.e. x_2,

III. $\gamma_{\Lambda_s n}(\mathbf{u}) = 0,\ \gamma_{\Lambda_s}(\tau_{ij})n_j t_i = 0,$

$\gamma_{\Lambda_o t}(\mathbf{u}) = 0,\ \gamma_{\Lambda_o}(\tau_{ij})n_j n_i = 0,$

$\gamma_{\Lambda_w}(\mathbf{u}) = 0$ (or $\gamma_{\Lambda_w n}(\mathbf{u}) = 0,\ \gamma_{\Lambda_w}(\tau_{ij})n_j t_i = 0$),

IV. $\gamma_\Lambda(\mathbf{u}) = 0$ (or as in III where slip is allowed).

Observe that the requirement $\mathbf{u} \in H^2(\Omega)^2$ is necessary for $\gamma(\tau_{ij})$ to be well-defined, but that for (3.30) it is sufficient to have $\mathbf{u} \in H^1_L(\Omega)^2$ where

(3.36) $H^1_L(\Omega) = \{u \in H^1(\Omega) \mid u_{,jj} \in L^2(\Omega)\}$.

With the notation as in section 3.2, let $1 \leq k \leq 4$ be fixed and suppose that $(\mathbf{u}, p)$ is a solution of the corresponding problem (3.30) - (3.35), denoted by $(\text{Aux})_k$. Then (3.30) implies that

$$e\int_\Omega u_j u_{i,j} v_i \, dx \; - \; \int_\Omega \tau_{ij,j} v_i \, dx \; = \; \int_\Omega f_i v_i \, dx \quad \forall\, \mathbf{v} \in L^2(\Omega)^2.$$

Since $p \in H^1(\Omega), \mathbf{u} \in H^2(\Omega)^2$ (or since $\mathbf{u} \in H^1_L(\Omega)^2$ and $\mathbf{u}$ satisfies (3.31)) and $\tau = \tau(\mathbf{u}, p)$ is symmetric by definition, it follows that $\tau \in \mathcal{T}$. Thus, the Green's formula (3.28) holds for τ and for every $\mathbf{v} \in V_k$. By the definition and symmetry of τ, this implies that

$$\text{(3.37)} \quad e\textstyle\int_\Omega u_j u_{i,j} v_i \, dx \; + \; \frac{2}{Re} \int_\Omega D_{ij}(\mathbf{u}) D_{ij}(\mathbf{v}) \, dx \; - \; \int_\Omega p v_{i,i} \, dx$$
$$= \; \textstyle\int_\Omega f_i v_i \, dx + \; \ll \pi_k(\tau(\mathbf{u}, p)), \; \gamma_k(\mathbf{v}) \gg_k \quad \forall \mathbf{v} \in V_k.$$

Equations (3.31) and (3.33) are equivalent to

$$\text{(3.38)} \quad (q, u_{i,i})_{0,\Omega} \; + (\mu, \gamma_{\Gamma n}(\mathbf{u}))_{\frac{1}{2},\Gamma} \; = \; 0 \;\forall\, (q, u) \in L^2(\Omega) \; \times \; H^{\frac{1}{2}}(\Gamma).$$

Moreover, (3.29) holds because $\tau_{ij} \in H^1(\Omega)$, $i, j = 1, 2$. Since $(A \times B)' = A' \times B'$ for all inner product spaces A and B, and (recall Proposition 3.3) the traces of τ_{ij} and $\mathbf{v}$ satisfy

$$\begin{aligned} \tau_{ij} n_j v_i \; &= (\tau \mathbf{n}) \cdot \mathbf{v} = \{(\tau \mathbf{n})_n \mathbf{n} + (\tau \mathbf{n})_t \mathbf{t}\} \cdot \{v_n \mathbf{n} + v_t \mathbf{t}\} \\ &= v_n (\tau \mathbf{n}) \cdot \mathbf{n} + v_t (\tau \mathbf{n}) \cdot \mathbf{t} \; = \tau_{ij} n_j (n_i v_n + t_i v_t) \end{aligned}$$

on the relevant portions of $\partial\Omega$, it is easy to see that for the respective problems (3.29) takes the form

(3.39) I. $\quad < \pi_n(\tau), \gamma_{\Gamma n}(\mathbf{v}) >_{\frac{1}{2},\Gamma} \; + \; < \pi_t(\tau), \gamma_{\Gamma t}(\mathbf{v}) >_{\frac{1}{2},\Gamma}$

$\quad = \int_\Gamma \gamma_\Gamma(\tau_{ij}) n_j \, (n_i \gamma_{\Gamma n}(\mathbf{v}) \; + \; t_i \gamma_{\Gamma t}(\mathbf{v})) \, ds \;\; \forall\, \mathbf{v} \in V_1,$

where $\pi_1 \; = \; (\pi_n, \pi_t)$ and $< \cdot, \cdot >_{\frac{1}{2},\Gamma}$ denotes duality pairing on $H^{\frac{1}{2}}(\Gamma)' \; \times \; H^{\frac{1}{2}}(\Gamma)$;

II. as in I, with $H_1^{\frac{1}{2}}(\Gamma)$ in place of $H^{\frac{1}{2}}(\Gamma)$, etc.;

III. $\quad < \pi_n^0(\tau), \gamma_{\Gamma n}^0(\mathbf{v}) >_{00,\Gamma} \; + \; < \pi^3(\tau), \; \left(\gamma_{\Gamma t}^A, \gamma_{\Lambda_o n}, \gamma_{\Lambda_s t}^D\right) \mathbf{v} >_3$

$\quad = \; \int_\Gamma \gamma_\Gamma(\tau_{ij}) n_j \left(n_i \gamma_{\Gamma n}^0(\mathbf{v}) \; + \; t_i \gamma_{\Gamma t}^A(\mathbf{v})\right) ds + \int_{\Lambda_o} \gamma_{\Lambda_o}(\tau_{ij}) n_j n_i \gamma_{\Lambda_o}(\mathbf{v}) ds$

$\quad + \; \int_{\Lambda_s} \gamma_{\Lambda s}(\tau_{ij}) n_j t_i \gamma_{\Lambda_s t}^D(\mathbf{v}) \, ds \;\forall\, \mathbf{v} \in V_3,$

where $\pi_3 = (\pi_n^0, \pi^3)$ and $< \cdot, \cdot >_3$ denotes duality pairing on $Y_3' \times Y_3$;

IV. as in I, with $H_{00}^{\frac{1}{2}}(\Gamma)$ in place of $H^{\frac{1}{2}}(\Gamma)$, etc..

Equations (3.34) and (3.35 III) imply that $\pi_t(\tau) = 0$ in I above, and similarly in II and IV, while $\pi^3(\tau) = 0$ in III. In the case of a slip condition at Λ_w in III, the operator π_n^0 cannot be "isolated" from π_3 as above in the Green's formula (cf. the remark following (3.12)). However, (3.34) and (3.35 III) imply that $\ll \pi_3(\tau), \gamma_3(\mathbf{v}) \gg_3$ depends only on $\gamma_{\Gamma n}(\mathbf{v})$. The situation is similar when a slip condition is applied at Λ_1 or Λ_2 in IV. Hence, in every case the last term in (3.37) reduces to the first term of the corresponding relation in (3.39). For convenience we shall henceforth denote the relevant trace space, inner product, duality pairing and trace operator by $N_k, [\cdot,\cdot]_k, < \cdot, \cdot >_k$ and γ_n^k respectively. As before, we shall sometimes omit the subscript k. Thus, by Riesz' theorem, there exists a unique $\lambda = \lambda(\mathbf{u}, p) \in N_k$ such that the last term in (3.37) equals $[\lambda, v_n]_k$, where v_n denotes $\gamma_n^k(\mathbf{v})$, for every $\mathbf{v} \in V_k$.

Let $\mathbf{u}^0 \in H^1(\Omega)^2$ be such that $\gamma(\mathbf{u}^0)$ satisfies conditions (3.32) and (3.35). The existence of $\mathbf{u}^0$ is assured if

(3.40) I. $\mathbf{u}_0 \in H^{\frac{1}{2}}(\Sigma)^2$;

II. $\mathbf{u}_0 \in H_1^{\frac{1}{2}}(\Sigma)^2$;

III. $u_{0t} \in H_{00}^{\frac{1}{2}}(\Sigma)$ and $u_{0n} \in H_E^{\frac{1}{2}}(\Sigma)$ $\left(u_{0n} \in H^{\frac{1}{2}}(\Sigma) \text{ if slip occurs at } \Lambda_w\right)$;

IV. $\mathbf{u}_0 \in H_{00}^{\frac{1}{2}}(\Sigma)^2$ (with suitable changes when slip is allowed at Λ).

In fact, if we denote the trace spaces listed above by U_0 and denote the corresponding closed subspaces $\{\mathbf{v} \in H^1(\Omega)^2 \mid \mathbf{v}$ satisfies (3.33) and (3.35) $\}$ by U^0, then it is not difficult to prove (by repeated application of the ideas and results of section 3.1 and the fact that Γ is bounded away from Σ) that $\gamma_\Sigma \in \mathcal{L}(U^0, U_0)$ and that this operator is surjective. Thus, by Lemma 3.2, it has a right inverse $\delta \in \mathcal{L}(U_0, U^0)$. Let $\mathbf{v} = \delta(\mathbf{u}_0)$. In the case of problems I, II and IV, it follows from the conditions on $\mathbf{u}_0$ below (2.12) that $\int_{\partial\Omega} \gamma_n(\mathbf{v})\, ds = 0$. In the case of problem III, $\mathbf{v}$ can be adjusted to also have this property. (Set $q = 0, r = 0$ in the proof of Lemma 4.4(c). This proves that the mapping $\gamma_\Sigma : \{\mathbf{v} \in U^0 \mid div\mathbf{v} = 0\} \rightarrow U_0$ is surjective, so that it has a bounded linear right inverse.) The important point is that $\|\mathbf{g}\|_{\frac{1}{2},\partial\Omega} \leq c(\Omega)\|\mathbf{u}_0\|_\Sigma$ where $\mathbf{g} = \gamma(\mathbf{v})$. It can be shown (cf. Girault and Raviart (1986), p.24) that there exists a $\mathbf{u}^0 \in H^1(\Omega)^2$ such that $div\mathbf{u}^0 = 0, \gamma(\mathbf{u}^0) = \mathbf{g}$ and $\|\mathbf{u}^0\|_1 \leq k(\Omega)\|\mathbf{g}\|_{\frac{1}{2}}$. Hence, $\mathbf{u}^0$ satisfies (3.31) - (3.33), (3.35) and $\|\mathbf{u}^0\|_1 \leq K(\Omega)\|\mathbf{u}_0\|_\Sigma$.

Let $\mathbf{w} = \mathbf{u} - \mathbf{u}^0$. Then $\mathbf{w} \in V_k$ since both $\mathbf{u}$ and $\mathbf{u}_0$ satisfy (3.32) and (3.35). Moreover, from (3.37) and the remarks above we get :

$$(3.41)\ \int_\Omega \left\{e(w_j w_{i,j} + w_j u^0_{i,j} + u^0_j w_{i,j})v_i + \tfrac{2}{Re} D_{ij}(\mathbf{w})D_{ij}(\mathbf{v})\right\} dx - \int_\Omega p\, v_{i,i}\, dx -$$

$$[\lambda(\mathbf{u}, p), v_n]_k = \int_\Omega \left\{(f_i - e u^0_j u^0_{i,j})v_i - \tfrac{2}{Re} D_{ij}(\mathbf{u}^0)D_{ij}(\mathbf{v})\right\} dx \ \forall\, \mathbf{v} \in V_k.$$

Since u_n, u^0_n and w_n belong to N_k ($\mathbf{u}, \mathbf{u}^0$ satisfy (3.35) and Γ is bounded away from Σ, so that it is always possible to construct functions in V_k that have the same boundary values on Γ), (3.38) is equivalent to

$$(3.42)\ (q, w_{i,i})_{0,\Omega} + [\mu, w_n]_k = -(q, u^0_{i,i})_{0,\Omega} - [\mu, u^0_n]_k \ \forall\, (q, \mu) \in L^2(\Omega) \times N_k.$$

On the basis of the discussion above, we may assume that the right hand side of (3.42) is identically zero.

Hence, we have shown that to every solution $(\mathbf{u}, p)$ of (Aux) there corresponds a solution $(\mathbf{w}, p, \lambda)$ of the <u>variational problem</u> given by :

(Var) Find $\mathbf{w} \in V_k$, $p \in L^2(\Omega)$ and $\lambda \in N_k$ which satisfy equations (3.41) and (3.42).

4 Existence and Uniqueness Results

In this chapter, results concerning the existence and uniqueness of solutions of problem (Var) will be established. First, a functional analytic framework is constructed (section 4.1) which is then applied to the linear (Stokes) form of problem (Var) (section 4.2). In similar fashion it is shown that the nonlinear (Navier-Stokes) problem can be analysed (section 4.4) via results obtained for a general class of nonlinear variational problems (section 4.3).

4.1 A class of linear problems

We shall follow the approach of Girault and Raviart (1986) (pp. 57 - 61) :

Let V be a Hilbert space with norm $\|\cdot\|_V$. Then the norm of its dual space V' is defined by

$$\|l\|_{V'} = \sup_{v \,\epsilon\, V} \frac{< l, v >}{\|v\|_V} \quad \forall\, l \,\epsilon\, V',$$

where $< \cdot, \cdot >$ denotes duality pairing on $V' \times V$. Whenever we write "$\sup\limits_{x \,\epsilon\, X}$" it is understood that the supremum is taken over all nonzero x in X. We define $M, \|\cdot\|_M, M'$ and $\|\cdot\|_{M'}$ in the same manner.

Let $a(\cdot,\cdot) : V \times V \to \Re$ and $b(\cdot,\cdot) : V \times M \to \Re$ be two continuous bilinear forms with norms

$$\|a\| = \sup_{u,v \,\epsilon\, V} \frac{a(u,v)}{\|u\|_V \|v\|_V}, \quad \|b\| = \sup_{v \,\epsilon\, V,\, m \,\epsilon\, M} \frac{b(v,m)}{\|v\|_V \|m\|_M}.$$

Our aim is to solve the following variational problem :

(VP) Given $k \,\epsilon\, M'$ and $l \,\epsilon\, V'$, find $w \,\epsilon\, V$ and $n \,\epsilon\, M$ such that

(4.1) $a(w,v) \;+\; b(v,n) \;=\; < l,v > \quad \forall\, v \in V,$

(4.2) $\qquad\qquad b(w,m) \;=\; < k,m > \quad \forall\, m \in M.$

Define the operators $\mathcal{A} \in \mathcal{L}(V,V')$ and $\mathcal{B} \in \mathcal{L}(V,M')$ by

(4.3) $< \mathcal{A}u, v > \;=\; a(u,v) \quad \forall\, u,\ v \in V,$

(4.4) $< \mathcal{B}v, m > \;=\; b(v,m) \quad \forall\, v \in V, \quad \forall\, m \in M.$

The *dual operator* $\mathcal{B}' \in \mathcal{L}(M,V')$ of $\mathcal{B}$ is defined by

(4.5) $< \mathcal{B}'m, v > \;=\; < \mathcal{B}v, m > \;=\; b(v,m) \quad \forall\, v \in V, \quad \forall\, m \in M.$

It is easy to show that $\|\mathcal{A}\|_{\mathcal{L}(V,V')} \;=\; \|a\|$ and $\|\mathcal{B}\|_{\mathcal{L}(V,M')} \;=\; \|\mathcal{B}'\|_{\mathcal{L}(M,V')} \;=\; \|b\|$. Moreover, equations (4.1) and (4.2) can now be written as

(4.6) $\mathcal{A}w \;+\; \mathcal{B}'n \;=\; l \;$ in V'.

(4.7) $\qquad\quad \mathcal{B}w \;=\; k \;$ in M'.

Problem (VP) is said to be *well-posed* if for every given pair (l,k), there exists a unique pair (w,n) satisfying (4.6) and (4.7), and if the resulting map $(l,k) \to (w,n)$ is continuous. This is equivalent to saying that the operator $\Upsilon \in \mathcal{L}(V \times M,\ V' \times M')$ defined by

(4.8) $\qquad \Upsilon(v,m) \;=\; (\mathcal{A}v + \mathcal{B}'m,\ \mathcal{B}v)$

is an isomorphism from $V \times M$ onto $V' \times M'$, i.e., Υ is bijective and its inverse is continuous.

Let $K = ker\mathcal{B} = \{v \in V \mid \mathcal{B}v = 0\}$. Then, as before, we define $K^{\perp} = \{v \in V \mid (u,v)_V = 0 \;\forall\, u \in K\}$ and $K^{a} = \left\{g \in V' \mid < g,v > = 0 \;\forall\, v \in K\right\}$. It is easily proved that $K, K^{\perp}$ and

K^a are closed subspaces of respectively V, V and V' and are therefore Hilbert spaces. Furthermore, we have the following important result.

Lemma 4.1 The following three properties are equivalent :

(a) there exists a constant $\beta > 0$ such that

(4.9) $\inf_{m \in M} \sup_{v \in V} \frac{b(v,m)}{\|v\|_V \|m\|_M} \geq \beta;$

(b) the operator $\mathcal{B}$ is an isomorphism form $K^{\perp}$ onto M' and

(4.10) $\|\mathcal{B}v\|_{M'} \geq \beta \|v\|_V \; \forall \, v \in K^{\perp};$

(c) the operator $\mathcal{B}'$ is an isomorphism from M onto K^a and

(4.11) $\|\mathcal{B}'m\|_{V'} \geq \beta \|m\|_M \; \forall \, m \in M.$

Proof Cf. Girualt and Raviart (1986), pp. 58 - 59. □

Lemma 4.1 was first proved by I. Babuška and F. Brezzi. O.A. Ladyzhenskaya proved a special case of the lemma in the context of the Navier-Stokes problem. Accordingly, we shall refer to (4.9) as the "LBB condition".

For every $g \in V'$, let Πg be defined as the restriction of g to K : $< \Pi g, v > = < g, v >$ $\forall \, v \in K$. Then $\|\Pi g\|_{K'} \leq \|g\|_{V'}$ and $\Pi \in \mathcal{L}(V', K')$.

Lemma 4.2 Suppose that the bilinear from $a(\cdot,\cdot)$ is *K-elliptic*, i.e., there exists a constant $\alpha > 0$ such that

(4.12) $a(v,v) \geq \alpha \|v\|_V^2 \; \forall \, v \in K.$

Then $\Pi\mathcal{A}$ is an isomorphism from K onto K' and $\|(\Pi\mathcal{A})^{-1}\|_{\mathcal{L}(K',K)} \leq \frac{1}{\alpha}$.

Proof The lemma follows directly from the Lax-Milgram theorem, a proof of which can be found in, e.g., Reddy (1986), pp. 117-120. □

Theorem 4.1 Let the notation be as before and assume that $a(\cdot,\cdot)$ is K-elliptic. Then problem (VP) is well-posed if and only if $b(\cdot,\cdot)$ satisfies the LBB condition.

Proof (1) Let $k \in M'$ and $l \in V'$ be given. By lemma 4.1 (b), there exists a unique $w_1 \in K^{\perp}$ such that $\mathcal{B}w_1 = k$ and $\|w_1\|_V \leq \|k\|_{M'}/\beta$.

By Lemma 4.2, there exists a unique $w_2 \in K$ such that $\Pi\mathcal{A}w_2 = \Pi(l - \mathcal{A}w_1)$ and $\|w_2\|_V \leq \frac{1}{\alpha}\|\Pi(l-\mathcal{A}w_1)\|_{K'} \leq \frac{1}{\alpha}\|l-\mathcal{A}w_1\|_{V'} \leq \frac{1}{\alpha}(\|l\|_{V'} + \|a\|\frac{1}{\beta}\|k\|_{M'})$.
Let $w = w_1 + w_2$. Then $\mathcal{B}w = k$, $\Pi\mathcal{A}w = \Pi l$ and

$$(4.13)\quad \|w\|_V \leq \|w_1\|_V + \|w_2\|_V \leq \tfrac{1}{\alpha}\|l\|_{V'} + \tfrac{1}{\beta}(1 + \tfrac{\|a\|}{\alpha})\|k\|_{M'}.$$

By definition of $\Pi, l - \mathcal{A}w \in K^a$. By Lemma 4.1 (c), there exists a unique $n \in N$ such that $\mathcal{B}'n = l - \mathcal{A}w$ and

$$(4.14)\quad \begin{aligned} \|n\|_M &\leq \tfrac{1}{\beta}\|l - \mathcal{A}w\|_{V'} \leq \tfrac{1}{\beta}(\|l\|_{V'} + \|a\|.\|w\|_V) \\ &\leq \tfrac{1}{\beta}(1 + \tfrac{\|a\|}{\alpha})(\|l\|_{V'} + \tfrac{\|a\|}{\beta}\|k\|_{M'}). \end{aligned}$$

Suppose that (w',n') is another solution of (VP). Set $w^* = w - w'$ and $n^* = n - n'$. Then

(i) $\quad a(w^*,v) + b(v,n^*) = 0 \;\forall\, v \in V$ and

(ii) $\quad b(w^*,m) = 0 \;\forall\, m \in M'$.

Thus $w^* \in K$ since $\mathcal{B}w^* = 0$ according to (ii). Choose $v = w^*$ and $m = n^*$. Then it follows from the K-ellipticity of $a(\cdot,\cdot)$ that $w^* = 0$. From (i) and (4.9) it follows that $n^* = 0$:

$$\|n^*\|_M \leq \sup_{v \in V} \frac{b(v,n^*)}{\beta\|v\|_V} = 0.$$

Thus, we have proved that there exists a unique solution $(w,n) \in V \times M$, which depends continuously on $(l,k) \in V' \times M'$.

(2). Conversely, assume that (VP) is well-posed. We shall show that condition (b) in Lemma 4.1 holds. Let $k \in M'$ and set $(w,n) = \Upsilon^{-1}(0,k)$. Then $k = \mathcal{B}w = \mathcal{B}u$ where u is the projection of w on $K^{\perp}$. If $v \in K^{\perp}$ and $\mathcal{B}v = 0$, then $v \in K \cap K^{\perp} = \{0\}$. Hence, $\mathcal{B}$ is a continuous linear bijection from $K^{\perp}$ onto M'. Furthermore, its inverse is bounded since $\|u\|_V \leq \|w\|_V \leq \|(w,n)\|_{V \times M} \leq \frac{1}{\beta}\|(0,k)\|_{V' \times M'} = \frac{1}{\beta}\|k\|_{V'}$ where $\frac{1}{\beta} = \|\Upsilon^{-1}\|_{\mathcal{L}(V' \times M', V \times M)}$. □

For dealing with specific problems it is convenient to introduce the operators $B = j_M^{-1} \cdot \mathcal{B} \in \mathcal{L}(V,M)$ and $B^{\times} = j_V^{-1} \cdot \mathcal{B}' \in \mathcal{L}(M,V)$. Here j_M and j_V are the Riesz isometries and consequently $B^{\times}$ is the *adjoint* of B:

$$(4.15)\quad (Bv,m)_M = (v, B^{\times}m)_V = b(v,m)\ \forall\, v \in V,\ \forall\, m \in M,$$

where $(\cdot,\cdot)_M$ and $(\cdot,\cdot)_V$ denote the inner products on M and V. It is clear that the algebraic and topological properties of B and $B^{\times}$ are identical to those of respectively $\mathcal{B}$ and $\mathcal{B}'$. Note also that $ker B = ker \mathcal{B} = K$.

<u>Theorem 4.2</u> (a) The LBB condition holds if and only if B is surjective, i.e., $Rg(B) = M$.

(b) Assume that $a(\cdot,\cdot)$ is K-elliptic and that $Rg(B)$ is closed. Then, for any given pair $(l,k) \in V' \times M'$, with $k \in (Rg(B)^{\perp})^{a}$, there exists a unique pair $(w,n) \in V \times Rg(B)$ such that $(w, n+n_0)$ is a solution of problem (VP) for every $n_0 \in Rg(B)^{\perp}$. If k does not satisfy the condition above, then (4.2) holds only for $m \in Rg(B)$. Moreover, the inequalities (4.13) and (4.14) again hold, so that the mapping $(l,k) \to (w,n)$ is continuous.

<u>Proof</u> (a) If the LBB condition holds, then $\mathcal{B}$ is surjective according to Lemma 4.1 (b). Conversely, suppose that $\mathcal{B}$ is surjective. Let $\mathcal{B}_0$ denote the restriction of $\mathcal{B}$ to $K^{\perp}$. As in part (2) of the proof of Theorem 4.1, it follows that $\mathcal{B}_0 \in \mathcal{L}(K^{\perp}, M')$ is bijective. The Open Mapping theorem now applies that $\mathcal{B}_0^{-1}$ is bounded. (This result is known as the Banach theorem). Thus, condition (b) of Lemma 4.1 holds with $\beta = \|\mathcal{B}_0^{-1}\|^{-1}_{\mathcal{L}(M',K^{\perp})}$. This proves statement (a) because B is surjective iff $\mathcal{B}$ is.

(b) It is easily verified that $Rg(B) = \{n \in M \mid$ there exists a $v \in V$ such that $(n,m)_M = b(v,m)\ \forall m \in M.\}$ Set $M_1 = Rg(B)$ and let $b_1(\cdot,\cdot)$ be the restriction of $b(\cdot,\cdot)$ to $V \times M_1$. Then the corresponding operator $B_1 \in \mathcal{L}(V, M_1)$ is surjective since $Rg(B_1) = \{n \in M_1 \mid$ for some $v \in V$, $(n,m)_M = b(v,m)\ \forall\, m \in M_1\} = M_1$. Hence, by (a) above, the LBB condition for $b_1(\cdot,\cdot)$ is satisfied.

Let $(l,k) \in V' \times M'$ be given. Then $k \in M_1'$ and thus it follows from Theorem 4.1 that there exists a unique pair $(w,n) \in V \times M_1$ such that

(i) $\quad a(w,v) + b(v,n) = \langle l, v \rangle \ \forall\, v \in V,$

(ii) $\quad\quad b(w,m) = \langle k, m \rangle \ \forall\, m \in M_1.$

Since $\|k\|_{M_1'} \leq \|k\|_{M'}$, the estimates (4.13) and (4.14) hold for (w,n). Furthermore, $b(v,n_0) = (Bv,n_0)_M = 0\ \forall\, v \in V$ and $\forall\, n_0 \in Rg(B)^{\perp}$, so that n may be replaced by $n + n_0$ in (i). Finally, $b(w,n_0) = (Bw,n_0)_M = 0\ \forall\, n_0 \in M_1^{\perp}$, so that M_1 may be replaced by M in (ii) iff $\langle k,m \rangle = 0\ \forall m \in M_1^{\perp}$. □

It can be proved, for every operator $T \in \mathcal{L}(V,M)$ and its adjoint $T^{\times}$, that $Rg(T)^{\perp} = ker T^{\times}$ and that $Rg(T)$ is closed iff $Rg(T^{\times})$ is closed. Thus, by Theorem 4.2 (a), the LBB condition holds iff $B^{\times}$ is injective, and in (b) we may require that $Rg(B^{\times})$ be closed instead.

Note that Theorem 4.2(b) can also be proved as follows : Set $M_2 = M/ker B^{\times}$ and define $b_2(\cdot,\cdot) : V \times M_2 \to \Re$ and $k_2 \in M_2'$ by $b_2(v,[m]) = b(v,n)$, $\langle k_2,[m] \rangle = \langle k,n \rangle$ $\forall\, v \in V$, $\forall\, [m] \in M_2$, where n denotes the projection of m on $Rg(B)$. Then proceed as in the proof above (cf. Oden and Carey (1983), pp. 101-110).

4.2 The Stokes problem

With the notation as in Chapter 3, let $1 \leq k \leq 4$ be fixed. To put the Stokes problem, i.e., problem (Var) with $e = 0$, into the framework of section 4.1 we set:

(4.16) $V = V_k$, $M = L^2(\Omega) \times N_k$ with inner products

$(\cdot,\cdot)_V = (\cdot,\cdot)_{1,\Omega}$, $((p,\lambda),(q,\tau))_M = (p,q)_{0,\Omega} + [\lambda,\tau]_k$

and norms $\|\cdot\|_V = \|\cdot\|_{1,\Omega}$, $\|(q,\tau)\|_M = \sqrt{((q,\tau),\ (q,\tau))_M}$,

$a(\mathbf{u},\mathbf{v}) = (2/Re)(D_{ij}(\mathbf{u}), D_{ij}(\mathbf{v}))_{0,\Omega}$,

$b(\mathbf{v},(q,\tau)) = -((q,\tau),\ (div\mathbf{v}, v_n))_M$ where $div\mathbf{v} = v_{i,i}$,

$< l, \mathbf{v} > = (\mathbf{f},\mathbf{v})_{0,\Omega} - a(\mathbf{u}^0,\mathbf{v})$,

$< k,(q,\tau) > = -b(\mathbf{u}^0,(q,\tau))$.

Here, $\mathbf{f} = (gL/U^2)\mathbf{d}$ where g is the gravitational acceleration and

$$\mathbf{d} = \begin{cases} (0,-1) & \text{in I, II} \\ \mathbf{0} & \text{in III} \\ (sin\theta_0,\ -\cos\theta_0) & \text{in IV.} \end{cases}$$

Henceforth the subscripts Ω and M will be omitted.

We know from Chapter 3 that $V_k, L^2(\Omega)$ and N_k, and therefore V and M, are Hilbert spaces. It is also easy to see that $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $< l,\cdot >$ and $< k,\cdot >$ are well-defined and linear in every argument. The boundedness of these operators is proved as follows:

$$| a(\mathbf{u},\mathbf{v}) | = (1/Re) \mid (u_{i,j} + u_{j,i}, D_{ij}(\mathbf{v}))_0 \mid = (1/Re) \mid (u_{i,j}, D_{ij}(\mathbf{v}) + D_{ji}(\mathbf{v}))_0 \mid$$
$$= (1/Re) \mid (u_{i,j}, v_{i,j} + v_{j,i})_0 \mid \leq (1/Re)\|u_{i,j}\|_0\|v_{i,j} + v_{j,i}\|_0$$
$$\leq (1/Re)\|u_{i,j}\|_0(\|v_{i,j}\|_0 + \|v_{j,i}\|_0) \leq \left(2\sqrt{3}/Re\right) \|\mathbf{u}\|_1\|\mathbf{v}\|_1 \quad \forall\, \mathbf{u},\mathbf{v} \in V_k$$

(using the Schwarz and triangle inequalities and the identity

$(2aA + (b+c)(B+C) + 2dD)^2 \leq 12(a^2 + ... + d^2)(A^2 + ... + D^2))$, so that $\|a\| \leq 2\sqrt{3}/Re$.

$| b(\mathbf{v},(q,\tau)) | = | ((q,\tau),(div\mathbf{v}, v_n)) | \leq c\|(q,\tau)\|$
with $c^2 = \|(div\mathbf{v}, v_n)\|^2 \leq (\|v_{1,1}\|_0 + \|v_{2,2}\|_0)^2 + \|v_n\|_k^2$
$\leq 2(\|v_{1,1}\|_0^2 + \|v_{2,2}\|_0^2) + t^2\|\mathbf{v}\|_1^2 \leq (2+t^2)\|\mathbf{v}\|_1^2$, where
$t = \|\gamma_n^k\|_{\mathcal{L}(V_k,N_k)} = t(\Omega)$, $\forall\, \mathbf{v} \in V_k$, $\forall\,(q,\tau) \in M$. Thus $\|b\| \leq \sqrt{2+t^2}$.

$|< l, \mathbf{v} >| \leq | (\mathbf{f},\mathbf{v})_0 | + | a(\mathbf{u}^0,\mathbf{v}) | \leq \|\mathbf{f}\|_0\|\mathbf{v}\|_0 + \|a\|.\|\mathbf{u}^0\|_1\|\mathbf{v}\|_1$
$\leq (\|\mathbf{f}\|_0 + \|a\|.\|\mathbf{u}^0\|_1)\|\mathbf{v}\|_1 \quad \forall\, \mathbf{v} \in V_k$, and therefore
$\|l\|_{V'} \leq \|\mathbf{f}\|_0 + \|\mathbf{u}^0\|_1\|a\|$. Similarly, $\|k\|_{M'} \leq \|\mathbf{u}^0\|_1\|b\|$.

Moreover, we may assume that $k = 0$ (see the paragraph below (3.40)). This is especially important for solving the nonlinear problem, as will become clear in the next section.

Clearly, the operator B defined in (4.15) is given by

$$(4.17) \quad B\mathbf{v} = -(div\mathbf{v}, v_n) \quad \forall\, \mathbf{v} \in V_k.$$

Thus $K = kerB = \{\mathbf{v} \in V_k \mid div\mathbf{v} = 0 \text{ in } \Omega,\ v_n = 0 \text{ on } \Gamma\} \subset V_k \subset V_0$
$= \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\}$, so that the K-ellipticity of $a(\cdot,\cdot)$ follows from the following lemma.

<u>Lemma 4.3</u> Let Ω be a bounded domain in $\Re^2$ with a Lipschitz continuous boundary. Then there exists a constant $\alpha' > 0$ such that

$$(4.18) \quad (D_{ij}(\mathbf{v}), D_{ij}(\mathbf{v}))_0 \geq \alpha'\|\mathbf{v}\|_1^2 \quad \forall\, \mathbf{v} \in V_0.$$

<u>Sketch of Proof</u> The lemma is deduced from one of Korn's inequalities:
If Ω satisfies the condition given above, then there exists a constant $C > 0$ such that

$$(\text{i}\ ) \quad | \mathbf{v} |_1^2 \leq C((D_{ij}(\mathbf{v}),\ D_{ij}(\mathbf{v}))_0 + \|\mathbf{v}\|_0^2) \quad \forall\, \mathbf{v} \in H^1(\Omega)^2.$$

The proof of this result is nontrivial and represents a major step in the analysis of a wide range of variational problems, including the ones considered here. Proofs can be found in Kikuchi and Oden (1988), pp. 104 - 109, and (for the case of a C^1 boundary) in Duvaut and Lions (1976), pp. 110 -115. Furthermore, there exists a constant $k > 0$ such that

(ii) $\quad (D_{ij}(\mathbf{v}),\ D_{ij}(\mathbf{v}))_0 \ \geq\ k\|\mathbf{v}\|_0^2 \quad \forall\, \mathbf{v} \in V_0.$

The proof of this result is by contradiction (so that k is unknown) and can be found on pp. 115 - 116 in either of the references given above. We know (see the comments following the definition of V_k in chapter 3) that there is a constant $c > 0$ such that

(iii) $\quad |\,\mathbf{v}\,|_1^2 \geq\ c\|\mathbf{v}\|_1^2 \quad \forall\, \mathbf{v} \in V_0.$

It follows from (i) - (iii) that (4.18) holds with $\alpha' = \frac{ck}{C(k+1)}$. □

Hence $a(\mathbf{v},\mathbf{v})\ \geq\ \alpha\|\mathbf{v}\|_1^2 \ \forall\, \mathbf{v} \in V_k$ with $\alpha = 2\alpha'/Re = A(\Omega)/Re.$

It remains to characterize $Rg(B)$ (to check if $b(\cdot,\cdot)$ satisfies the LBB condition).

Lemma 4.4 (a) The operator div maps $H_0^1(\Omega)^2$ onto the space $L_0^2(\Omega)\ =\ \{q \in L^2(\Omega) \mid \int_\Omega q\ dx\ =\ 0\}.$

(b) For problems I, II and IV we have

$$(4.19)\quad Rg(B)\ =\ S\ =\ \{(q,\tau) \in M \mid \int_\Omega q\ dx\ =\ \int_\Gamma \tau\ ds\}.$$

This is a closed subspace of M and its orthogonal complement is given by

$$(4.20)\quad Rg(B)^\perp\ =\ span\,\{(1,\tau^*)\}\ =\ \{k(1,\tau^*) \mid k \in \Re\}$$

where $\tau^* \in N_k$ is such that $[\tau^*,\tau]_k\ =\ -\int_\Gamma \tau\ ds \quad \forall\, \tau \in N_k.$

(c) In the case of problem III, B is surjective.

Proof (a) Define $W = \{\mathbf{w} \in H_0^1(\Omega)^2 \mid div\mathbf{w} = 0\}$ and let $W^\perp$ be the orthogonal complement of W in $H_0^1(\Omega)^2$ with respect to the inner product $(v_{i,j}, w_{i,j})_0$ associated with the norm $|\cdot|_1$. Then it can be proved that the operator div is an isomorphism from $W^\perp$ onto $L_0^2(\Omega)$ (cf. Temam (1978), pp. 15, 32, or Girault and Raviart (1986), p. 24).

(b) (1) By definition of V_k, $\gamma_n(\mathbf{v}) = 0$ on $\partial\Omega \setminus \Gamma$ $\forall\, \mathbf{v} \in V_1$ and $\forall\, \mathbf{v} \in V_4$. Moreover, $\int_{\Lambda_0 \cup \Lambda_1} \gamma_n(\mathbf{v})\, ds = 0$ $\forall\, \mathbf{v} \in V_2$ due to the periodicity condition.

Thus if $(q,\tau) = B\mathbf{v}$ for some $\mathbf{v} \in V_k$, then it follows via the standard Green's formula for functions in $H^1(\Omega)$ that

$$\int_\Omega q\, dx = \int_\Omega 1.v_{i,i}\, dx = \int_{\partial\Omega} \gamma(1.v_i)n_i\, ds = \int_\Gamma v_n\, ds = \int_\Gamma \tau\, ds.$$

Conversely, suppose that $(q,\tau) \in M$ satisfies the compatibility condition $\int_\Omega q\, dx = \int_\Gamma \tau\, ds$. Since γ_n^k is surjective, there exists a $\mathbf{v} \in V_k$ such that $\gamma_n^k(\mathbf{v}) = \tau$. As above, it follows that

$$\int_\Omega div\mathbf{v}\, dx = \int_\Gamma v_n\, ds = \int_\Gamma \tau\, ds = \int_\Omega q\, dx.$$

Hence $q - div\mathbf{v} \in L_0^2(\Omega)$. By (a), there exists a $\mathbf{w} \in H_0^1(\Omega)^2 \subset V_k$ such that $div\mathbf{w} = q - div\mathbf{v}$. Set $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Then $\mathbf{u} \in V_k$ and $B\mathbf{u} = (q,\tau)$. Thus $Rg(B) = S$.

(2) Assume that $(q,\tau) \in M$ and that there exists a sequence (q_n,τ_n) in S such that $(q_n,\tau_n) \to (q,\tau)$ in M. Then $q_n \to q$ in $L^2(\Omega)$ and $\tau_n \to \tau$ in N_k, so that

$$\begin{aligned}
&\left| \textstyle\int_\Omega q\, dx - \int_\Gamma \tau\, ds \right| = \left| \textstyle\int_\Omega (q - q_n) dx + \int_\Gamma (\tau_n - \tau) ds \right| \quad (\text{since } (q_n,\tau_n) \in S) \\
&\le |(1, q - q_n)_0| + |(1, \tau_n - \tau)_{0,\Gamma}| \le \|1\|_0 \|q - q_n\|_0 + \|1\|_{0,\Gamma} \|\tau_n - \tau\|_{0,\Gamma} \\
&\le \|1\|_0 \|q - q_n\|_0 + \|1\|_{0,\Gamma} \|\tau_n - \tau\|_k \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\int_\Omega q\, dx = \int_\Gamma \tau\, ds$, i.e., $(q,\tau) \in S$. Hence S is closed in M.

(3) By definition, $S^\perp = \{(p,\lambda) \in M \mid (p,q)_0 + [\lambda,\tau]_k = 0 \ \forall\, (q,\tau) \in M$ satisfying $\int_\Omega q\, dx = \int_\Gamma \tau\, ds\}$. Let $(p,\lambda) \in S^\perp$. Then $(p,q)_0 = (p,q)_0 + [\lambda, 0]_k = 0$ $\forall\, q \in L_0^2(\Omega)$, i.e.,

$p \in L_0^2(\Omega)^\perp$. But $L_0^2(\Omega) = \{q \in L^2(\Omega) \mid (k,q)_0 = 0 \;\; \forall\, k \in \Re\} = \Re^\perp$ (where $\Re$ is understood to represent the subspace of a.e. constant functions in $L^2(\Omega)$) and $\Re^{\perp\perp} = \Re$ since $\Re$ is a finite - dimensional, and therefore closed, subspace of $L^2(\Omega)$. Thus $p \in \Re$.

Suppose that $p = 0$. Then $[\lambda, \tau]_k = (0, q_\tau)_0 + [\lambda, \tau]_k = 0 \;\; \forall\, \tau \in N_k$ (with q_τ chosen so that $(q_\tau, \tau) \in S$), i.e., $\lambda = 0$. Since $S^\perp \neq \{(0,0)\}$ (from (4.19)), it follows that there exists a pair $(p, \tau) \in S^\perp$ with $p \neq 0$.

Suppose that $Q = span\,\{(p_i, \tau_i) \mid i \in J\} \subset S^\perp$, with J some index set. By the arguments above, we may assume that $0 \neq p_i \in \Re \;\; \forall\, i \in J$. Thus $Q = span\,\{(1, \tau_i/p_i) \mid i \in J\}$. Now, $(1, \tau^*) \in S^\perp$ iff $(1,q)_0 + [\tau^*, \tau]_k = 0 \;\; \forall\, (q, \tau) \in M$ such that $(1,q)_0 = (1,\tau)_{0,\Gamma}$, i.e., iff $[\tau^*, \tau]_k = -(1,\tau)_{0,\Gamma} \;\; \forall\, \tau \in N_k$. By the Riesz representation theorem, this equation has a unique solution $\tau^* \in N_k$ since the right hand side defines an element of N_k'. Hence $Q = span\,\{(1, \tau^*)\}$. Since $S^\perp$ is a linear space, it follows that $S^\perp \subset \overline{Q} = Q$.

Conversely, let $k \in \Re$ and set $(p, \lambda) = k(1, \tau^*)$. Then $(p,q)_0 + [\lambda, \tau]_k = k(1,q)_0 - k(1,\tau)_{0,\Gamma} = 0 \;\; \forall\, (q,\tau) \in S$. Thus $S^\perp = span\,\{(1,\tau^*)\}$.

(c) Let $(q, \tau) \in L^2(\Omega) \times N_3$ be given. Since γ_n^3 is surjective, there exists a $\mathbf{v}^0 \in V_3$ such that $\gamma_n^3(\mathbf{v}^0) = \tau$. If $\int_{\Lambda_o} v_n^0 \, ds = 0$, set $\mathbf{v} = \mathbf{v}^0$. If $\int_{\Lambda_o} v_n^0 \, ds \neq 0$, then we can define $\Lambda_p \subset \Lambda_o$, with meas$(\Lambda_p) > 0$ and $\overline{\Lambda_p} \subset \Lambda_o$, such that $\int_{\Lambda_p} v_n^0 \, ds \neq 0$. Thus $\int_{\Lambda_p} \phi v_n^0 \, ds \neq 0$ for some smooth function ϕ on Λ_o with $\phi = 0$ on $\Lambda_o \setminus \Lambda_p$ (else $v_n^0 = 0$ on Λ_p, since $\mathcal{D}(\Lambda_p)$ is dense in $L^2(\Lambda_p)$). Now define

$$\boldsymbol{\phi} = \begin{cases} \gamma(\mathbf{v}^0) & \text{on } \partial\Omega \setminus \Lambda_o \\ (1 + k\phi)\gamma(\mathbf{v}^0) & \text{on } \Lambda_o \end{cases}$$

where $k = (\int_\Omega q \, dx - \int_\Gamma \tau \, ds - \int_{\Lambda_o} v_n^0 \, ds) / \int_{\Lambda_o} \phi v_n^0 \, ds$.

Then $\boldsymbol{\phi} \in H^{\frac{1}{2}}(\partial\Omega)^2$ and therefore $\gamma(\mathbf{v}) = \boldsymbol{\phi}$ for some $\mathbf{v} \in H^1(\Omega)^2$. From the definition of $\boldsymbol{\phi}$ and V_3 it follows that $\mathbf{v} \in V_3$. Thus we have constructed a $\mathbf{v} \in V_3$ such that

$$\int_\Omega div\mathbf{v}\,dx = \int_{\Gamma\cup A_o} \gamma_n(\mathbf{v})ds = \int_\Omega q\,dx.$$

As in the last part of (b)(1), it now follows that $(q,\tau) \in Rg(B)$. □

Note that this proof also applies when slip conditions are used in problems III and IV since the operators γ_n^k corresponding to these situations are again surjective.

From Theorem 4.1 and 4.2 we now obtain:

<u>Theorem 4.3</u> Assume that Ω and $\mathbf{u}_0$ satisfy the conditions set out in chapters 2 and 3. Then we have:

For each of problems I, II and IV, there exist uniquely determined functions $\mathbf{w} \in V_k$ and $(p,\lambda) \in S$ such that the solution set of problem (Var) is given by $\{(\mathbf{w}, p+c, \lambda + c\tau^*) \mid c \in \Re\}$, where S and τ^* are as in Lemma 4.4.

In the case of problem III, (Var) has a unique solution $(\mathbf{w},p,\lambda) \in V_3 \times L^2(\Omega) \times N_3$.

Moreover, for all four problems, there exists a constant C which depends only on Ω so that

$$(4.21)\quad \|\mathbf{w}\|_1 \le C(Re.gL/U^2 + \|\mathbf{u}_0\|_\Sigma),$$

$$\|p\|_0, \|\lambda\|_k \le C(gL/U^2 + \|\mathbf{u}_0\|_\Sigma/Re). \ \square$$

Note that if we set $U = \nu/L$, then $Re = 1$ and $gL/U^2 = gL^3/\nu^2$ in (4.21).

The nonuniqueness of the Lagrange multipliers in the case of problems I, II and IV reflects the fact that the pressure field is determined only up to a constant by problem (Aux). In problem III it is unique due to (3.35 III).

It remains to consider the relation between problems (Aux) and (Var). In section 3.3 it was shown that to every solution $(\mathbf{u},p) \in H^2(\Omega) \times H^1(\Omega)$ of (Aux) there corresponds a uniquely determined solution $(\mathbf{w},p,\lambda)$ of (Var) for every given $\mathbf{u}^0$.

Let $\mathbf{u}_0$ be fixed. Let $\mathbf{u}^0$ be any extension to Ω of $\mathbf{u}_0$ of the type constructed in chapter 3. Assume that the corresponding problem (Var) has at least one solution and let $(\mathbf{w}, p, \lambda)$ be any of these with $(p, \lambda) \in S$. Let $\mathbf{u} = \mathbf{w} + \mathbf{u}^0$. Then $(\mathbf{u}, p, \lambda)$ satisfies

(i) $a(\mathbf{u}, \mathbf{v}) - (p, div\mathbf{v})_0 - [\lambda, v_n]_k = (\mathbf{f}, \mathbf{v})_0 \;\; \forall\, \mathbf{v} \in V_k,$

(ii) $(q, div\mathbf{u})_0 + [\tau, u_n]_k = 0 \;\; \forall\, q \in L^2(\Omega), \;\; \forall\, \tau \in N_k.$

Let $\mathbf{u}^{0'}, \mathbf{w}', \mathbf{p}', \lambda'$ and $\mathbf{u}'$ be defined in the same manner. Then $(\mathbf{u}^*, p^*, \lambda^*) = (\mathbf{u} - \mathbf{u}', p - p', \lambda - \lambda')$ satisfies these equations with $\mathbf{f}$ replaced by $\mathbf{0}$. Furthermore, $\mathbf{u}^* \in V_k$ and thus $u_n^* \in N_k$. With $q = div\mathbf{u}^*$ and $\tau = u_n^*$, it follows from (ii) that $div\mathbf{u}^* = 0$ and $u_n^* = 0$. With $\mathbf{v} = \mathbf{u}^*$ in (i), this implies that $\alpha\|\mathbf{u}^*\|_1^2 \leq a(\mathbf{u}^*, \mathbf{u}^*) = 0$, i.e., $\mathbf{u}^* = 0$. From (i) it now follows that $p^* = 0$ and $\lambda^* = 0$ (since $(p^*, \lambda^*) \in Rg(B)$ by definition of (p, λ) and (p', λ')). This proves that $(\mathbf{u}, p, \lambda)$ is independent of the choice of $\mathbf{u}^0$ for a given $\mathbf{u}_0$.

Furthermore, from (4.21) it follows that
$\|\mathbf{u}\|_1 \leq \|\mathbf{u}^0\|_1 + \|\mathbf{w}\|_1 \leq C'(\Omega)(gL^3/\nu^2 + \|\mathbf{u}_0\|_\Sigma).$

Since $\mathbf{u}^0$ satisfies (3.31) - (3.33) and $\mathbf{w} \in V_k$, it follows from (3.42) that $\mathbf{u}$ satisfies (3.31) - (3.33). Similarly, $\mathbf{u}$ satisfies the kinematic boundary conditions in (3.35). The remaining equations only make sense if greater regularity is assigned to $\mathbf{u}$ and $\mathbf{p}$.

Assume that $\mathbf{u} \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$. Then, by reversing the steps used in section 3.3, (3.41) becomes

$$(4.22) \;\; -\int_\Omega \tau_{ij,j} v_i dx + \int_{\partial\Omega} \gamma(\tau_{ij}) n_j \gamma(v_i) ds - [\lambda, v_n]_k$$

$$= \int_\Omega f_i v_i \, dx \;\; \forall\, \mathbf{v} \in V_k.$$

Thus, for every $\mathbf{v} \in \mathcal{D}(\Omega)^2$, $-(\tau_{ij,j}, v_i)_0 = (f_i, v_i)_0$. Since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, this implies that (3.30) is satisfied. Therefore, (4.22) reduces to

$$(4.23) \;\; [\lambda, v_n]_k = (\tau_n, v_n)_{0,\partial\Omega} + (\tau_t, v_t)_{0,\partial\Omega} \;\; \forall\, \mathbf{v} \in V_k,$$

where $\tau_n = \gamma(\tau_{ij})n_jn_i$ and $\tau_t = \gamma(\tau_{ij})n_jt_i$. It is easy to see from the definitions of the spaces V_k that the right hand side of (4.23) is equal to the following:

I. $(\tau_n, v_n)_{0,\Gamma} + (\tau_t, v_t)_{0,\Gamma} = N + T$, say;

II. $N + T + \int_{\Lambda_0 \cup \Lambda_1} \gamma(\tau_{ij})n_j\gamma(v_i)\, ds$;

III. $N + T + (\tau_n, v_n)_{0,\Lambda_o} + (\tau_t, v_t)_{0,\Lambda_s}$;

IV. $N + T$.

Recall that the trace operators $\gamma_k : V_k \to Z_k$, defined in (3.12), are surjective. Thus, for problem I it follows from (4.23) that $0 = 0 + \tau_t, \tau)_{0,\Gamma} \;\; \forall\, \tau \in H^{\frac{1}{2}}(\Gamma)$. Since $\mathcal{D}(\Gamma)$ (and therefore $H^{\frac{1}{2}}(\Gamma)$) is dense in $L^2(\Gamma)$, this implies that $\tau_t = 0$ on Γ. An identical argument shows that (3.34) is also satisfied in problem IV.

If for problem II we assume that p is 1-periodic in x_1, then τ_{ij} is too (since $\mathbf{u}$ is) and consequently the integral above vanishes. Moreover, $(\tau_n, \tau_t) \in Z_2$ and so it follows in similar fashion as above that (3.34) holds.

For problem III it follows from (4.23) that

$(\tau_t, u)_{0,\Gamma} + (\tau_n, v)_{0,\Lambda_o} + (\tau_t, w)_{0,\Lambda_s} = 0 \;\forall\, (u, v, w) \in Y_3$.

Since $\mathcal{D}(\Gamma) \times \mathcal{D}(\Lambda_o) \times \mathcal{D}(\Lambda_s) \subset Y_3$ and $\mathcal{D}(\cdot)$ is dense in $L^2(\cdot)$, it follows easily that both (3.34) and (3.35 III) are satisfied. Similar arguments apply whenever slip conditions are used in problems III or IV.

In summary, if a solution of (Var) satisfies the assumptions above, then $(\mathbf{u}, p)$ solves (Aux). Moreover, (4.23) now becomes

(4.24) $[\lambda, v_n]_k = (\tau_n, v_n)_{0,\Gamma} \;\; \forall\, \mathbf{v} \in V_k$.

Thus, we have a functional representation of the function $\tau_n\,|_\Gamma$, which appears in the important equation (2.16).

In conclusion, let us briefly consider the relation between the Stokes problem and the Signorini problem described in section 2.1. We shall not give the classical formulation of this problem or give the exact definitions of the quantities involved. This can be found in the references given below. The relevant fact is that the Signorini problem can be posed as a mixed variational problem of the following form:

Find $(\mathbf{u}, p, \lambda) \in V \times Q \times N$ such that

$a(\mathbf{u}, \mathbf{v}) - (p, div\mathbf{v}) - [\lambda, v_n] = < f, \mathbf{v} > \quad \forall\, \mathbf{v} \in V,$

$(q, div\mathbf{v}) = 0 \quad \forall\, q \in Q,$

$[\tau - \lambda, v_n - g] \geq 0 \quad \forall\, \tau \in N.$

Here, $V = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_\Sigma(\mathbf{v}) = \mathbf{0}\}$ is the space of admissible displacements, $Q = L^2(\Omega), N = \left\{\tau \in H^{\frac{1}{2}}(\Gamma) \mid \tau \leq 0\right\}$, f is a functional involving external body forces on Ω and surface tractions on Λ, g is the initial gap between Ω and F, and $\mathbf{u}$ is the displacement field. Note that the hydrostatic pressure p and contact pressure λ are employed as Lagrange multipliers for the constraints

$div\mathbf{u} = 0$ in Ω, $v_n - g \leq 0$ on Γ.

The associated operator B and LBB condition are identical to that of problems I - IV. The interesting feature is that, as with Λ_o in problem III, there is a portion of $\partial\Omega \setminus \Gamma$ on which the constraint $\gamma_n(\mathbf{v}) = 0$ is not enforced, namely Λ. By exploiting this "free" variable it can be proved that the LBB condition holds. When the boundary is of class $C^{2,1}$, it is possible to give a constructive proof, i.e., an explicit lower bound on the constant β can be established (cf. Kikuchi and Oden (1988), pp. 177 - 179). For the case of a Lipschitz domain, two proofs are given in Oden, Kikuchi and Song (1980), pp. 56 - 60, both of which are based on the same strategy as the proof of Lemma 4.4(c).

4.3 A class of nonlinear problems

In this section we shall study a nonlinear generalization of problem (VP).

Let the notation be as in section 4.1. Thus, V and M are Hilbert spaces and $b(\cdot,\cdot) : V \times M \to R$ is a continuous bilinear form. Furthermore, let the form

$$(4.25)\quad a(\cdot\,;\cdot,\cdot) : V \times V \times V \to \Re$$

be such that, for every $w \in V$, the mapping $a(w;\cdot,\cdot) : V \times V \to \Re$ is a continuous bilinear form. We shall consider the following nonlinear variational problem :

(NVP) Given $l \in V'$, find $w \in V$ and $n \in M$ which satisfy

$$(4.26)\quad a(w;w,v) + b(v,n) = < l, v > \quad \forall\, v \in V,$$

$$(4.27)\quad b(w,m) = 0 \quad \forall\, m \in M.$$

Let the operator $B \in \mathcal{L}(V,V')$ be defined as in section 4.1. For every $w \in V$, the operator $\mathcal{A}(w) \in \mathcal{L}(V,V')$ is defined by

$$(4.28)\quad < \mathcal{A}(w)u, v > = a(w;u,v) \quad \forall\, u,v \in V.$$

Then equations (4.31) and (4.32) can be written as

$$(4.29)\quad \mathcal{A}(w)w + \mathcal{B}'n = l \text{ in } V',$$

$$(4.30)\quad \mathcal{B}w = 0 \text{ in } M'.$$

The spaces $K = ker\mathcal{B}, K^{\perp}, K^{a}$ and the operator $\Pi \in \mathcal{L}(V',K')$ are defined as in section 4.1. If problem (NVP) is compared to problem (VP), then it becomes clear from the proof of Theorem 4.1 that the crucial issue is to derive an appropriate generalization of the Lax - Milgram theorem. To be precise, we must find conditions on $a(\cdot\,;\cdot,\cdot)$ sufficient for solving the following nonlinear problem :

(N) Given $l \in V'$, find $w \in K$ such that

(4.31) $a(w;w,v) \ = \ < l, v > \quad \forall\, v \in K,$

or equivalently such that

(4.32) $\Pi \mathcal{A}(w)w \ = \ \Pi l$ in K'.

<u>Theorem 4.4</u> (a) Assume that the following conditions hold :

(i) the space K is separable, i.e., K has a countable dense subset ;

(ii) there exists a constant $\alpha > 0$ such that

(4.33) $a(v;v,v) \ \geq \alpha \|v\|_V^2 \ \ \forall\, v \in K;$

(iii) for each $v \in K$, the mapping $u \rightarrow a(u;u,v)$ is sequentially weakly continuous on K, i.e.,

(4.34) $\lim\limits_{n\to\infty} \ < h, u_n > = < h, u > \quad \forall\, h \in K'$ (u_n converges weakly to u in K) implies
$\lim\limits_{n\to\infty} a(u_n;u_n,v) \ = \ a(u;u,v) \ \ \forall\, v \in K.$

Then problem (N) has at least one solution $w \in K$. Moreover, every solution satisfies the estimate

(4.35) $\|w\|_V \ \leq \ \|\Pi l\|_{K'}/\alpha \ \ \leq \ \|l\|_{V'}/\alpha.$

(b) Assume that conditions (i) and (iii) are satisfied and that (4.33) holds $\forall\, v \in S_d = \{v \in K \mid \ \|v\|_K \ \leq \ d\}$ for some fixed $d > 0$. Then, for every $l \in V'$ with

(4.36) $\|\Pi l\|_{K'} \ \leq \ \alpha d,$

problem (N) has at least one solution $w \in S_d$, for which (4.35) holds.

Proof A proof of part (a) is given in Girault and Raviart (1986), pp. 279 - 281. In effect, the following more general result is proved :

Let K be a Hilbert space and $a(\cdot,\cdot) : V \times V \rightarrow \Re$ a mapping such that

(i) K is separable;

(ii) $a(v,v) \geq \alpha\|v\|_K^2 \ \forall\, v \in K$ for some $\alpha > 0$;

(iii) for each $v \in K$, the map $u \rightarrow a(u,v)$ is sequentially weakly continuous on K;

(iv) for each $u \in K$, the map $v \rightarrow a(u,v)$ is linear and continuous on K.

Then, given $h \in K'$, there exists at least one element $w \in K$ such that
$a(w,v) = < h,v > \ \forall\, v \in K$.

For every solution w, $\|w\|_K \leq \|h\|_{K'}/\alpha$ since

$$\alpha\|w\|_K^2 \leq a(w,w) = < h,w > \leq \|h\|_{K'}\|w\|_K.$$

We shall now give the proof of this result and show how it can be adjusted to establish the analogue of (b). These alterations will be given in square brackets. To obtain Theorem 4.4, define $a(\cdot,\cdot)$ by $a(u,v) = a(u;u,v)$ and set $h = \Pi l$.

(Note that if it is assumed that, for each $v \in K$, the map $u \rightarrow a(u,v)$ is linear and continuous, then the hypothesis (iii) above also holds (since every $g \in K'$ is weakly sequentially continuous; see (4.34)). Hence, for separable K, the result above is a direct generalization of the Lax-Milgram theorem.)

(1) We shall need the following result, which can be derived from the Brouwer fixed point theorem:

Let H be a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)$ and corresponding norm $|\cdot|$. Let F be a continuous mapping from H into H with the following property: there exists a constant $r > 0$ such that
$(F(h),h) \geq 0$ for all $h \in H$ with $| h | = r$.

Then there exist an $h \in H$ such that $F(h) = 0$ and $| h | \leq r$.

(2) Using (1), we now construct a sequence of approximate solutions via the Galerkin method. Since the space K is separable, there exists a linearly independent subset $S = \{v_n \mid n \geq 1\}$ of K such that $span(S)$ is dense in K. For each $n \geq 1$, let K_n denote $span(\{v_1, ..., v_n\})$ and consider the following problem:

(N_n) Find $w_n \in K_n$ such that $a(w_n, v) = < h, v > \ \forall\, v \in K_n$.

Define the mapping $F_n : K_n \rightarrow K_n$ by
$(F_n(v), v_i)_K = a(v, v_i) - < h, v_i >, \ 1 \leq i \leq n.$

Then $w_n \in K_n$ is a solution of problem (N_n) iff $F_n(w_n) = 0$. Clearly, $(F_n(v), v)_K = a(v, v) - < h, v > \ \forall\, v \in K_n$, so that from hypothesis (ii) we obtain:

$$(F_n(v), v)_K \geq (\alpha\|v\|_K - \|h\|_{K'})\|v\|_K \ \forall\, v \in K_n \ [\forall\, v \in K_n \cap S_d].$$

Hence, choosing $r = \|h\|_{K'}/\alpha$, we get for all $v \in K_n$ with $\|v\|_K = r$ that $(F_n(v), v)_K \geq 0$ [because $r \leq d$ when $\|h\|_{K'} \leq \alpha d$]. Moreover, F_n is continuous in K_n by virtue of hypothesis (iii). Since K_n is finite-dimensional, we may apply (1): there exists at least one solution w_n of problem (N_n) [such that $\|w_n\|_K \leq r$].

If w_n is any solution of problem (N_n) [and if $w_n \in S_d$], then it follows from the inequality following the definition of F_n that $\|w_n\|_K \leq r$.

(3) Construct a sequence (w_n) in K by taking, for each n, one arbitrary solution of problem (N_n) [out of those solutions generated by (1)]. Then this sequence is bounded in K. Since K is a Hilbert space and therefore reflexive, it follows from a well-known theorem that we can extract a subsequence (w_m) and some $w \in K$ such that $w_m \rightarrow w$ weakly in K. Now hypothesis (iii) implies that

$$\lim_{m \rightarrow \infty} a(w_m, v) = a(w, v) \ \forall\, v \in K.$$

For an arbitrary v_i, it follows from (N_n) that $a(w_m, v_i) = \langle h, v_i \rangle \quad \forall\, m \geq i$, which implies that $a(w,v) = \langle h, v \rangle \quad \forall\, v \in S$ and so $\forall\, v \in span(S)$. Since $span(S)$ is dense in K, it follows from (iv) and the continuity of h that the equation holds $\forall\, v \in K$. □

<u>Theorem 4.5</u> (a) Assume that

(i) the bilinear form $a(u;\cdot,\cdot)$ is uniformly K-elliptic with respect to u, i.e., there exists a constant $\alpha > 0$ such that

(4.37) $a(u;v,v) \geq \alpha\|v\|_K^2 \quad \forall\, u, v \in K;$

(ii) the mapping $u \to \Pi\mathcal{A}(u) : K \to \mathcal{L}(K,K')$ is locally Lipschitz continuous, i.e., there exists a continuous increasing function $L : \Re^+ \to \Re^+$ such that for all $r > 0$

$$(4.38) \quad \sup_{v,w \in K} \frac{|a(u_1;v,w) - a(u_2;v,w)|}{\|v\|_V \|w\|_V} = \|\Pi\mathcal{A}(u_1) - \Pi\mathcal{A}(u_2)\|_{L(K,K')}$$

$$\leq L(r)\|u_1 - u_2\|_V \quad \forall\, u_1, u_2 \in S_r = \{u \in K \mid \|u\|_V \leq r\}.$$

Then, for every $l \in V'$ satisfying

(4.39) $(\|\Pi l\|_{K'}/\alpha^2) L(\|\Pi l\|_{K'}/\alpha) < 1,$

problem (N) has a unique solution, for which the estimate (4.35) holds.

(b) Assume that condition (ii) is satisfied and that (4.37) holds for all $u \in S_d$ for some fixed $d > 0$. Then, for every $l \in V'$ satisfying (4.36) and (4.39), problem (N) has a solution in S_d, which is unique in S_d and satisfies (4.35).

<u>Proof</u> As in the proof of Theorem 4.4, we shall now show how the proof of (a) given in Girault and Raviart (1986), p. 282, can be adapted to establish (b).

From (4.37) and the Lax-Milgram theorem (see proof of Theorem 4.1) it follows that, for each $v \in K$ $[v \in S_d]$, the operator $\Pi\mathcal{A}(v) \in \mathcal{L}(K,K')$ is bijective and $T(v) = (\Pi\mathcal{A}(v))^{-1}$ belongs to $X = \mathcal{L}(K',K)$ with $\|T(v)\|_X \leq 1/\alpha$. Equation (4.32) can now be written as

$w = F(w)$ in K, where $F(v) \; = \; T(v)\Pi l \;\; \forall \, v \in K$.

We shall now show that F is a contraction on S_r, where $r = \;\; \|\Pi l\|_{K'}/\alpha > 0$. (If $\|\Pi l\|_{K'} = 0$, then $w = 0$.) For all $u, v \in S_r$ [since $r \leq d$ when (4.36) holds],

$$
\begin{aligned}
T(u) - T(v) &= T(u)[\Pi\mathcal{A}(v)\cdot T(v)] \; - \; [T(u)\cdot\Pi\mathcal{A}(u)]T(v) \\
&= T(u)[\Pi\mathcal{A}(v) - \Pi\mathcal{A}(u)]T(v)
\end{aligned}
$$

so that, from (4.38),

$$
\begin{aligned}
\|T(u) - T(v)\|_X &\leq (1/\alpha)\|\Pi\mathcal{A}(v) - \Pi\mathcal{A}(v)\|_{\mathcal{L}(K,K')}\cdot(1/\alpha) \\
&\leq L(r)\|u - v\|_V/\alpha^2
\end{aligned}
$$

and thus, by (4.39),

$$
\begin{aligned}
\|F(u) - F(v)\|_V &= \|(T(u) - T(u))\Pi l\|_V \\
&\leq L(r)\|u - v\|_V\|\Pi l\|_{K'}/\alpha^2 = \; C\;\|u - v\|_V \text{with} C > 1
\end{aligned}
$$

Since S_r is closed and therefore complete, it follows that F has a unique fixed point in S_r, say w. By definition of S_r, w satisfies (4.35).

For every $v \in K$ [$v \in S_d$], $F(v) \in S_r$ because

$$
\|F(v)\|_V \;\leq\; \|T(v)\|_X\|\Pi l\|_{K'} \;\leq\; \|\Pi l\|_{K'}/\alpha \;=\; r.
$$

Thus, if $w' \in K$ [$w' \in S_d$] is any fixed point of F, then $w' = F(w') \in S_r$ and hence $w' = w$ according to the previous paragraph. This proves that the solution $w \in S_r$ of problem (N) is unique in K [in S_d]. □

It is easily verified that if (w, n) is a solution of problem (NVP), then w is a solution of problem (N). As in the linear case, the converse statement holds when the LBB condition is satisfied :

Theorem 4.6 (a) Assume that $b(\cdot,\cdot)$ satisfies that LBB condition. Then, for each solution w of problem (N), there exists a unique $n \in M$ such that (w, n) is a solution of problem (NVP).

(b) Let $B \in \mathcal{L}(V, M)$ be defined as in (4.15) and assume that $Rg(B)$ is closed. Then, for each solution w of problem (N), there exists a unique $n \in Rg(B)$ such that $(w, n+m)$ is a solution of problem (NVP) for every $m \in Rg(B)^{\perp}$.

Proof (a) For a given l, assume that w is a solution of problem (N). Then $Bw = 0$ since $w \in K$. Furthermore, $\Pi(l - \mathcal{A}(w)w) = 0$ and thus $l - \mathcal{A}(w)w \in K^a$. By Lemma 4.1(c), there exists a unique $n \in M$ such that $B'n = l - \mathcal{A}(w)w$. (Note that the approach used in proving Theorem 4.1(a) fails here for nonzero $k \in M'$.)

(b) This follows immediately via the arguments used in the proofs of Theorem 4.1(b) and (a) above. □

It remains to derive general *a priori* estimates for the solution(s) of problem (NVP). Let (w, n) be any solution.

(i) If $a(\cdot\,;\cdot,\cdot)$ satisfies (4.33), then it follows from

$\alpha\|w\|_V^2 \leq a(w; w, w) = < l, w > \leq \|\Pi l\|_{K'}\|w\|_V$

that w satisfies (4.35).

(ii) If $b(\cdot,\cdot)$ satisfies that LBB condition, or $Rg(B)$ is closed and $n \in Rg(B)$, then it follows from (4.11) that

$$(4.40) \quad \|n\|_M \leq \|l - \mathcal{A}(w)w\|_{V'}/\beta \leq (\|l\|_{V'} + \|\mathcal{A}(w)w\|_{V'})/\beta$$

$$\leq \left(\|l\|_{V'} + \|\mathcal{A}(w)\|_{\mathcal{L}(V,V')}\|w\|_V\right)/\beta.$$

If we also assume that $a(\cdot;\cdot,\cdot) : V \times V \times V \to \Re$ is a trilinear continuous form with norm $\|a\|$, then $\mathcal{A} \in \mathcal{L}(V, \mathcal{L}(V, V'))$ and $\|\mathcal{A}\| \leq \|a\|$. Hence, when (i) applies,

$$(4.41) \quad \|n\|_M \leq (\|l\|_{V'} + \|a\|.\|w\|_V^2)/\beta \leq \|l\|_{V'}(1 + \|l\|_{V'}\|a\|/\alpha^2)/\beta.$$

4.4 The Navier-Stokes problem

In the present section, we shall apply the results of section 4.3 to the Navier-Stokes problem, i.e., problem (Var) with $e = 1$.

With the notation as before, let $1 \leq k \leq 4$ be fixed. To put problem $(\mathrm{Var})_k$ within the framework of section 4.3, we set:

(4.42) $V = V_k,\ M = L^2(\Omega) \times N_k$

with inner products $(\cdot,\cdot)_1, (\cdot,\cdot)$ and norms $\|\cdot\|_1, \|\cdot\|$ as in (4.16),

$a(\mathbf{w}; \mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{w}, \mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}^0, \mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}^0, \mathbf{v})$

with $a_0(\mathbf{u}, \mathbf{v}) = \frac{2}{Re}(D_{ij}(\mathbf{u}), D_{ij}(\mathbf{v}))_0$

and $a_1((\mathbf{w}, \mathbf{u}, \mathbf{v}) = \int_\Omega w_j u_{i,j} v_i \, dx$,

$b(\mathbf{v}, (q, \tau)) = -((q, \tau), (div\mathbf{v}, v_n))$,

$< l, \mathbf{v} > = (\mathbf{f}, \mathbf{v})_0 - a_0(\mathbf{u}^0, \mathbf{v}) - a_1(\mathbf{u}^0, \mathbf{u}^0, \mathbf{v})$.

It is clear that the spaces V, M and the forms $a_0(\cdot,\cdot), b(\cdot,\cdot)$ are precisely those used in section 4.2, so that their properties are known. As before,

$$K = \left\{ \mathbf{v} \in V_k \mid div\mathbf{v} = 0,\ \gamma_n^k(\mathbf{v}) = 0 \right\}.$$

It remains to determine the properties of $a_1(\cdot,\cdot,\cdot), a(\cdot;\cdot,\cdot)$ and $< l, \cdot >$.

<u>Lemma 4.5</u> (a) Let Ω be a bounded domain in $\Re^2$ with a Lipschitz continuous boundary. Then the form $a_1(\cdot,\cdot,\cdot)$ is well-defined, trilinear and continuous on $(H^1(\Omega)^2)^3$.

(b) Let Ω be as above and let Λ be a (possibly empty) portion of $\partial\Omega$. Let $\mathbf{v}, \mathbf{w} \in H^1(\Omega)^2$ with $div\mathbf{w} = 0$ and $\gamma(\mathbf{w}) \cdot \mathbf{n}|_{\partial\Omega\setminus\Lambda} = 0$. Then

(4.43) $a_1(\mathbf{w}, \mathbf{v}, \mathbf{v}) = (1/2) \int_\Lambda \gamma(w_j) n_j \gamma(v_i) \gamma(v_i) ds.$

(c) Let $\mathbf{w} \in K$. Then, for problems I – IV, we have

(4.44) $a_1(\mathbf{w},\mathbf{v},\mathbf{v}) = b \ \forall \, \mathbf{v} \in H$,

(4.45) $a_1(\mathbf{w},\mathbf{u},\mathbf{v}) + a_1(\mathbf{w},\mathbf{v},\mathbf{u}) = c \ \forall \, \mathbf{u},\mathbf{v} \in H$,

where b, c and H are defined respectively by

I, IV	$H = H^1(\Omega)^2, \ b = c = 0;$
II	$H = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_{\Lambda_0}(\mathbf{v})(x_2) = \gamma_{\Lambda_1}(\mathbf{v})(x_2)$ for a.e.$x_2\}$,
	$b = c = 0;$
III	$H = \{\mathbf{v} \in H^1(\Omega)^2 \mid \gamma_{\Lambda_o t}(\mathbf{v}) = 0\}, b = (1/2)\int_{\Lambda_o} \gamma(w_1)\gamma(v_1)^2 ds,$
	$c = \int_{\Lambda_o} \gamma(w_1)\gamma(u_1)\gamma(v_1) ds.$

(d) For every $\mathbf{v} \in H$ (with H as in (c)), the mapping $\mathbf{u} \rightarrow a_1(\mathbf{u},\mathbf{u},\mathbf{v})$ is sequentially weakly continuous on K.

<u>Proof</u> (a) According to the Sobolev imbedding theorem (cf., e.g., Adams (1975)), the space $H^1(\Omega)$ is continuously imbedded in $L^4(\Omega)$, i.e., $H^1(\Omega) \subset L^4(\Omega)$ and there exists a constant $c = c(\Omega)$ such that $\|h\|_{0,4} \leq d\|h\|_1 \ \forall \, h \in H^1(\Omega)$. (As usual, $L^4(\Omega)$ denotes the space of (cosets of) functions $h : \Omega \rightarrow \Re$ such that $| \, h \, |^4$ is Lebesgue-integrable, and is equipped with the norm

$$\|h\|_{0,4,\Omega} = \left(\int_\Omega | \, h \, |^4 \, dx\right)^{\frac{1}{4}} = \sqrt{\|h^2\|_{0,\Omega}}.)$$

Thus, if $\mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega)^2$, then $v_i, w_j \in L^4(\Omega)$, so that $v_i^2, w_j^2 \in L^2(\Omega)$. By Hölder's inequality,

$$\int_\Omega (v_i w_j)^2 \, dx = \int_\Omega v_i^2 w_j^2 \, dx \leq \|v_i^2\|_0 \|w_j^2\|_0 < \infty$$

and thus $v_i w_j \in L^2(\Omega)$. Thus, again by Hölders inequality,

$$| \, a_1(\mathbf{w},\mathbf{u},\mathbf{v}) \, | = | \, (u_{i,j}, v_i w_j)_0 \, | \leq \|u_{i,j}\|_0 \|v_i w_j\|_0$$

$$\leq \|u_{i,j}\|_0\sqrt{\|v_i^2\|_0\|w_j^2\|_0} = \|u_{i,j}\|_0\|v_i\|_{0,4}\|w_j\|_{0,4}$$

$$\leq c^2\|u_{i,j}\|_0\|v_i\|_1\|w_j\|_1 \leq 4c^2 \mid \mathbf{u} \mid_1 \|\mathbf{v}\|_1\|\mathbf{w}\|_1 < \infty.$$

Hence, $a_1(\cdot,\cdot,\cdot)$ is well-defined and bounded, with norm $\|a_1\| \leq 4c^2$. The linearity is obvious.

This proof is given in Ladyzhenskaya (1963), p. 96, and Girault and Raviart (1986), p. 284. Generalizations of the result can be found in Temam (1983), pp. 12-13.

(b) An important property of the Sobolev spaces is that the smooth functions are dense in them. Specifically, the space $\mathcal{D}(\overline{\Omega}) = \{h\,|_\Omega \mid h \in \mathcal{D}(\Re^2)\}$ is dense in $H^1(\Omega)$. Let $\mathbf{w}$ be as given in the lemma. Then, for every $\mathbf{v} \in \mathcal{D}(\overline{\Omega})^2$, it follows from the standard Green's formula that

$$\begin{aligned} a_1(\mathbf{w},\mathbf{v},\mathbf{v}) &= \int_\Omega w_j v_{i,j} v_i dx = (1/2)\int_\Omega w_j(v_i v_i),j\ dx \\ &= (1/2)\int_{\partial\Omega} \gamma(w_j)v_i v_i n_j ds - (1/2)\int_\Omega w_{j,j} v_i v_i\ dx \\ &= (1/2)\int_\Lambda \gamma(w_j)n_j v_i v_i\ ds = a_2(\mathbf{w},\mathbf{v}), \text{ say.} \end{aligned}$$

Let $\mathbf{v} \in H^1(\Omega)^2$. Then there exists a sequences $(\mathbf{v}^n)$ in $\mathcal{D}(\overline{\Omega})^2$ such that $\mathbf{v}^n \to \mathbf{v}$ in $H^1(\Omega)^2$. Moreover, every $\mathbf{v}^n$ satisfies the equation above. From (a) and the boundedness of $\|\mathbf{v}^n\|_1$ (since every (weakly or strongly) convergent sequence is bounded), it follows that (with no summation over n) :

$$\mid a_1(\mathbf{w},\mathbf{v},\mathbf{v}) - a_1(\mathbf{w},\mathbf{v}^n,\mathbf{v}^n) \mid \leq \mid a_1(\mathbf{w},\mathbf{v},\mathbf{v}-\mathbf{v}^n) \mid + \mid a_1(\mathbf{w},\mathbf{v}-\mathbf{v}^n,\mathbf{v}^n) \mid$$

$$\leq \|a_1\|.\|\mathbf{w}\|_1(\|\mathbf{v}\|_1 + \|\mathbf{v}^n\|_1)\|\mathbf{v}-\mathbf{v}^n\|_1 \to 0 \text{ as } n \to \infty.$$

According to the Sobolev-imbedding theorem, $H^{\frac{1}{2}}(\Lambda)$ is continuously imbedded in $L^4(\Lambda)$ (cf. Grisvard (1985), pp. 27, 25, 37). Let $K = K(\Lambda)$ be the imbedding constant. Then

$$\begin{aligned} \mid a_2(\mathbf{w},\mathbf{v}) \mid \leq m\|\gamma(v_i)\gamma(v_i)\|_{0,\Lambda} &= m\Big(\|\gamma(v_1)\|_{0,4,\Lambda}^2 + \|\gamma(v_2)\|_{0,4,\Lambda}^2\Big) \\ &\leq K^2 m\|\gamma(\mathbf{v})\|_{\frac{1}{2},\Lambda}^2 < \infty \end{aligned}$$

where $m = \|\gamma(w_j)n_j\|_{0,\Lambda}/2$. Furthermore,

$$
\begin{aligned}
| a_2(\mathbf{w},\mathbf{v}) - a_2(\mathbf{w},\mathbf{v}^n) | &\leq \sum_{i=1}^{2} m\|\gamma(v_i)^2 - \gamma(v_i^n)^2\|_{0,\Lambda} \\
= \sum_{i=1}^{2} m\sqrt{(\gamma(v_i - v_i^n)^2, \gamma(v_i + v_i^n)^2)_{0,\Lambda}} &\leq \sum_{i=1}^{2} m\|\gamma(v_i - v_i^n)\|_{0,4\Lambda}\|\gamma(v_i + v_i^n)\|_{0,4,\Lambda} \\
\leq \sum_{i=1}^{2} K^2 m\|\gamma(v_i - v_i^n)\|_{\frac{1}{2},\Lambda}\|\gamma(v_i + v_i^n)\|_{\frac{1}{2},\Lambda} &\leq \sum_{i=1}^{2} C\|v_i - v_i^n\|_1\|v_i + v_i^n\|_1 \\
\leq 2C\|\mathbf{v} - \mathbf{v}^n\|_1(\|\mathbf{v}\|_1 + \|\mathbf{v}^n\|_1) &\rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}
$$

Thus, we have proved that

$$
a_1(\mathbf{w},\mathbf{v},\mathbf{v}) = \lim_{n\to\infty} a_1(\mathbf{w},\mathbf{v}^n,\mathbf{v}^n) = \lim_{n\to\infty} a_2(\mathbf{w},\mathbf{v}^n) = a_2(\mathbf{w},\mathbf{v}).
$$

(c) By the definition of K (and V_k), if $\mathbf{w} \in K$ then $div\mathbf{w} = 0$ and

I, IV $\gamma(\mathbf{w}) \cdot \mathbf{n} = 0$ on $\partial\Omega$ so that (4.44) follows immediately from (b) (with Λ an empty set);

II $\gamma(\mathbf{w}) \cdot \mathbf{n} = 0$ on $\Gamma \cup \Sigma$, so that (4.44) follows from (b) (with $\Lambda = \Lambda_0 \cup \Lambda_1$) and the periodicity of $\mathbf{v}$ and $\mathbf{w}$;

III $\gamma(\mathbf{w}) \cdot \mathbf{n} = 0$ on $\partial\Omega \setminus \Lambda_o$, so that (4.44) follows from (b) (with $\Lambda = \Lambda_o$) and the definitions of V_3 and H.

To obtain (4.45), we use (4.44) to evaluate $a_1(\mathbf{w},\mathbf{u}+\mathbf{v},\mathbf{u}+\mathbf{v})$, expand the left hand side via the multilinear properties of $a_1(\cdot,\cdot,\cdot)$ and then apply (4.44) to $a_1(\mathbf{w},\mathbf{u},\mathbf{u})$ and $a_1(\mathbf{w},\mathbf{v},\mathbf{v})$.

(d) Let $\mathbf{u} \in K$ and let $(\mathbf{u}^n)$ be a sequence in K such that $\mathbf{u}^n \rightarrow \mathbf{u}$ weakly in K. Then $\mathbf{u}^n \rightarrow \mathbf{u}$ weakly in $H^1(\Omega)^2$ because $(H^1(\Omega)^2)' \subset K'$. By the Sobolev imbedding theorem, $H^1(\Omega)$ is compactly imbedded into $L^2(\Omega)$, which implies that $\mathbf{u}^n \rightarrow \mathbf{u}$ strongly in $L^2(\Omega)^2$.

For each of the problems I - IV, the space $\mathcal{H} = H \cap D(\overline{\Omega})^2$ is dense in H. (This is true for problem II if the periodic smooth functions are dense in the periodic Sobolev functions.

We shall not attempt to prove this, but shall assume that it is true). Let $\mathbf{v} \in \mathcal{H}$. Since $K \subset H$, it follows from (4.45) that

$$a_1(\mathbf{u},\mathbf{u},\mathbf{v}) = c(\mathbf{u},\mathbf{v}) - a_1(\mathbf{u},\mathbf{v},\mathbf{u}),$$

$$a_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}) = c(\mathbf{u}^n,\mathbf{v}) - a_1(\mathbf{u}^n,\mathbf{v},\mathbf{u}^n) \;\; \forall \; n,$$

where

$c(\mathbf{u},\mathbf{v}) = \int_{\Lambda_o} \gamma(u_1)^2\gamma(v_1)\, ds$ in problem III and $c(\cdot,\cdot) = 0$ for the other problems. Since $\mathbf{v}$ is smooth, $| v_i |$ and $| v_{i,j} |$ are bounded on $\overline{\Omega}$ by some constant, say d. Hence,

$$\begin{aligned}
&| a_1(\mathbf{u},\mathbf{v},\mathbf{u}) - a_1(\mathbf{u}^n,\mathbf{v},\mathbf{u}^n) | \leq \int_\Omega | v_{i,j} | \cdot | u_iu_j - u_i^nu_j^n | \; dx \\
\leq & \sum_{i,j=1}^{2} d\int_\Omega (| u_i - u_i^n | \cdot | u_j | + | u_i^n | \cdot | u_j - u_j^n |)\, dx \\
\leq & \sum_{i,j=1}^{2} d\Big(\|u_i - u_i^n\|_0\|u_j\|_0 + \|u_i^n\|_0\|u_j - u_j^n\|_0\Big) \\
\leq & \; 4d(\|\mathbf{u}\|_0 + \|\mathbf{u}^n\|_0)\|\mathbf{u} - \mathbf{u}^n\|_0 \; \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

If $T \in H^{\frac{1}{2}}(\Lambda_o)'$, then $T \cdot \gamma_{\Lambda_o} \in K'$ and thus $T(\gamma_{\Lambda_o}(\mathbf{u}^n)) \rightarrow T(\gamma_{\Lambda_o}(\mathbf{u}))$. Hence, $\gamma_{\Lambda_o}(\mathbf{u}^n) \rightarrow \gamma_{\Lambda_o}(\mathbf{u})$ weakly in $H^{\frac{1}{2}}(\Lambda_o)$, and therefore strongly in $L^2(\Lambda_o)$, since $H^{\frac{1}{2}}(\Lambda_o)$ is compactly imbedded in $L^2(\Lambda_o)$. It now follows (for problem III) in formally the same manner as above that

$$\begin{aligned}
| c(\mathbf{u},\mathbf{v}) - c(\mathbf{u}^n,\mathbf{v}) | \quad &\leq \quad 4d(\|\gamma(\mathbf{u})\|_{0,\Lambda} + \|\gamma(\mathbf{u}^n)\|_{0,\Lambda})\|\gamma(\mathbf{u}) - \gamma(\mathbf{u}^n)\|_{0,\Lambda} \\
&\rightarrow \quad 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Thus $\lim_{n\rightarrow\infty} a_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}) \;=\; a_1(\mathbf{u},\mathbf{u},\mathbf{v}) \;\; \forall \, \mathbf{v} \in \mathcal{H}$.

Let $\mathbf{v} \in H$ and let $(\mathbf{v}^m)$ be a sequence in $\mathcal{H}$ such that $\mathbf{v}^m \rightarrow \mathbf{v}$ in $H^1(\Omega)^2$. Let $s = \sup_n \{\|\mathbf{u}^n\|_1, \|\mathbf{u}\|_1\} < \infty$. Then, for every m and n,

$$\begin{aligned}
&| a_1(\mathbf{u},\mathbf{u},\mathbf{v}) - a_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}) | \\
\leq & \; | a_1(\mathbf{u},\mathbf{u},\mathbf{v} - \mathbf{v}^m) | + A_{mn} + | a_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}^m - \mathbf{v}) | \\
\leq & \; 2s^2\|a_1\| \cdot \|\mathbf{v} - \mathbf{v}^m\|_1 + A_{mn}
\end{aligned}$$

where $A_{mn} = | a_1(\mathbf{u},\mathbf{u},\mathbf{v}^m) - a_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}^m) |$.

Let $\varepsilon > 0$ be given. Choose M so that $\|\mathbf{v} - \mathbf{v}^M\|_1 < \varepsilon/(4s^2\|a_1\|)$. Since $\mathbf{v}^M \in \mathcal{H}$, we can choose N so that $A_{Mn} < \varepsilon/2 \; \forall\, n \geq N$. Hence, $| a_1(\mathbf{u},\mathbf{u},\mathbf{v}) - a_1(\mathbf{u}^n,\mathbf{u}^n,\mathbf{v}) | < \varepsilon$ $\forall\, n \geq N$. Thus, the equation above holds for every $\mathbf{v} \in H$. □

It follows from (a) and the properties of $a_0(\cdot,\cdot)$ that $a(\cdot;\cdot,\cdot)$ is well-defined on $V_k \times V_k \times V_k$ and that the mapping $(u,v) \to a(w;u,v)$ is a continuous bilinear form on $V_k \times V_k$ for every $w \in V_k$. In fact, for every $\mathbf{w} \in V_k$ we have
$| a(\mathbf{w};\mathbf{u},\mathbf{v}) | \leq (\|a_0\| + \|a_1\|.\|\mathbf{w}\|_1 + 2\|a_1\|.\|\mathbf{u}^0\|_1)\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \;\; \forall\, \mathbf{u},\mathbf{v} \in V_k$,
so that
$\|\mathcal{A}(\mathbf{w})\| \leq \|a_0\| + \|a_1\|(2\|\mathbf{u}^0\|_1 + \|\mathbf{w}\|_1)$.

Similarly, $l \in V_k'$ with $\|l\|_{V_k'} \leq \|\mathbf{f}\|_0 + (\|a_0\| + \|a_1\|.\|\mathbf{u}^0\|_1)\|\mathbf{u}^0\|_1$.

It is clear from (a) that, for every $\mathbf{v}$ in K, the mapping $\mathbf{u} \to a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{u}^0,\mathbf{u},\mathbf{v}) + a_1(\mathbf{u},\mathbf{u}^0,\mathbf{v})$ belongs to K'. Using (d), this implies that the mapping $\mathbf{u} \to a(\mathbf{u},\mathbf{u},\mathbf{v})$ is sequentially weakly continuous on K for every $\mathbf{v}$ in K.

Furthermore, for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}$ and $\mathbf{w}$ in V_k,

$$\begin{aligned} | a(\mathbf{u}_1,\mathbf{v},\mathbf{w}) - a(\mathbf{u}_2,\mathbf{v},\mathbf{w}) | &= | a_1(\mathbf{u}_1 - \mathbf{u}_2;\mathbf{v},\mathbf{w}) | \\ &\leq \|a_1\|.\|\mathbf{u}_1 - \mathbf{u}_2\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1. \end{aligned}$$

Hence, condition (4.38) is satisfied with $L(r) = \|a_1\|$ for all r, i.e., $\Pi\mathcal{A}$ is uniformly Lipschitz continuous in K.

For problems I, II and IV, it follows from (c) that $a_1(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \;\; \forall\, \mathbf{u} \in K, \; \forall\, \mathbf{v} \in V_k$. Moreover, from the properties of $\mathbf{u}^0$ and the proof of (c) it is clear that $a_1(\mathbf{u}^0,\mathbf{v},\mathbf{v}) = 0 \;\; \forall\, \mathbf{v} \in V_k$ (since $\gamma_\Sigma(\mathbf{v}) = \mathbf{0}$). Therefore,

$$\begin{aligned} a(\mathbf{u},\mathbf{v},\mathbf{v}) &= a_0(\mathbf{v},\mathbf{v}) + a_1(\mathbf{v},\mathbf{u}^0,\mathbf{v}) \\ &\geq (\alpha_0 - \|a_1\|.\|\mathbf{u}^0\|_1)\|\mathbf{v}\|_1^2 \;\; \forall\, \mathbf{u},\mathbf{v} \in K. \end{aligned}$$

Since $\|\mathbf{u}^0\|_1 \leq K(\Omega)\|\mathbf{u}_0\|_\Sigma$, it follows that the hypotheses (4.33) and (4.37) are satisfied, with $\alpha = \alpha_0 - \|a_1\|.\|\mathbf{u}^0\|_1$, when $\|\mathbf{u}_0\|_\Sigma$ is sufficiently small. In the case of problem III,

$$\begin{aligned} a(\mathbf{u},\mathbf{v},\mathbf{v}) &= a_0(\mathbf{v},\mathbf{v}) + a_1(\mathbf{u},\mathbf{v},\mathbf{v}) + a_1(\mathbf{u}^0,\mathbf{v},\mathbf{v}) + a_1(\mathbf{v},\mathbf{u}^0,\mathbf{v}) \\ &\geq (\alpha_0 - \|a_1\|(\|\mathbf{u}\|_1 + 2\|\mathbf{u}^0\|_1))\|\mathbf{v}\|_1^2 \ \forall\, \mathbf{u},\mathbf{v} \in V_k. \end{aligned}$$

Let c and d be arbitrary positive constants such that $0 < 2c + d < \alpha_0/\|a_1\|$. Then (4.33) holds for all $\mathbf{v} \in S_d$ and (4.37) holds for all $\mathbf{u} \in S_d$ and $\mathbf{v} \in K$, with $\alpha = \alpha_0 - (2c + d)\|a_1\|$, if $\|\mathbf{u}^0\|_1 \leq c$.

Finally, since the Sobolev spaces are separable, K is too.

It was proved in chapter 3 that it is always possible to define $\mathbf{u}^0$ such that $\|\mathbf{u}^0\|_1 \leq K\|\mathbf{u}_0\|_\Sigma$, where K is a constant which depends only on Ω. Choose $U = \nu/L$. Then it follows from the estimates derived earlier that
$\|a_0\| \leq 2\sqrt{3} < 4$, $\|a_1\| \leq C(\Omega)$, $\|\mathbf{f}\|_0 \leq gm/\nu^2$ with $m(\Omega) = \sqrt{meas(\Omega)/L^3} = \sqrt{Vol}/L^3$,
and therefore
$\|l\|_{V'} \leq gm/\nu^2 + (4 + CKu)Ku$, where u denotes $\|\mathbf{u}_0\|_\Sigma$. Furthermore, we may take $\alpha_0 = A(\Omega)$.

For problems I, II and IV, a sufficient condition for (4.33) and (4.37) to hold is therefore that

(4.46) $u < A/CK$.

Then we may set $\alpha = A - CKu > 0$, so that a sufficient condition for (4.39) to hold is that

$gm/\nu^2 + (4 + CKu)Ku < (A - CKu)^2/C$, or equivalently,

(4.47) $gm/\nu^2 + (4 + 2A)Ku < A^2/C$.

It is easy to see that (4.47) implies (4.46). For (4.33) and (4.37) to be satisfied with respect to S_d in the case of problem III, it is sufficient that

(4.48) $d + 2Ku < A/C.$

Then $\alpha = A - (d + 2Ku)C > 0$, so that for (4.39) to hold it is sufficient that

(4.50) $(4 + CKu)Ku < (A - (d + 2Ku)C)^2/C,$

and for (4.36) to hold it is sufficient that

(4.51) $(4 + CKu)Ku < d(A - (d + 2Ku)C).$

From the results of this section and theorems 4.4 - 4.6 we can derive the following:

Theorem 4.7 Assume that Ω and $\mathbf{u}_0$ satisfy the regularity requirements, etc. specified in chapters 2 and 3.

(a) Then, for each of problems I, II and IV, there exist constants C_1, C_2 and C_3 (with $C_3 < C_1$) which depend only on Ω such that problem (NVP) has at least one solution $\mathbf{w} \in K$ when $\|\mathbf{u}_0\|_\Sigma < C_1$. Moreover, $\mathbf{w}$ is unique if $C_2 g/\nu^2 + \|\mathbf{u}_0\|_\Sigma < C_3$.

For every solution $\mathbf{w}$ of problem (NVP), there exists a unique pair $(p, \lambda) \in S$ such that $(\mathbf{w}, p + c, \lambda + c\tau^*)$ is a solution of problem (Var) for every $c \in \Re$, with S and τ^* as in (4.20). Every solution of problem (Var) is generated in this manner.

(b) In the case of problem III, there exists constants C_4 and C_5 which depend only on Ω such that problem (NVP) has at least one solution $\mathbf{w} \in S_d$ when $d + C_4\|\mathbf{u}_0\|_\Sigma < C_5$. Furthermore, for every $d < C_5$ there exists a constant $C_6 = C_6(\Omega, d)$ such that $\mathbf{w}$ is unique in S_d when $\|\mathbf{u}_0\|_\Sigma < C_6$.

For every solution $\mathbf{w}$ of problem (NVP) there exists a unique pair (p, λ) such that $(\mathbf{w}, p, \lambda)$ is a solution of problem (Var). □

Let $(\mathbf{w}, p, \lambda)$ be as above. For problems I, II and IV, it follows from (4.35) that when (4.46) holds, then

$$(4.52)\quad \begin{aligned} \|\mathbf{w}\|_1 &\le (gm/\nu^2 + (4 + CKu)Ku)/(A - CKu) \\ &\le C_7(\Omega)(g/\nu^2 + u) \text{ for sufficiently small } u. \end{aligned}$$

Since $\|\mathcal{A}(\mathbf{w})\mathbf{u}\|_{V_k'} \le (\|a_0\| + \|a_1\|(\|\mathbf{w}\|_1 + 2\|\mathbf{u}^0\|_1))\|\mathbf{u}\|_1 \quad \forall\, \mathbf{u}, \mathbf{w}, \in V_k$,
it follows from (4.40) and (4.52) that

$$(4.53)\quad \begin{aligned} \|p\|_0, \|\lambda\|_k &\le (gm/\nu^2 + (\|\mathbf{w}\|_1 + Ku)[4 + C(\|\mathbf{w}\|_1 + Ku)])/\beta \\ &\le C_8(\Omega)(g/\nu^2 + u) \text{ for sufficiently small } g/\nu^2, u. \end{aligned}$$

In the case of problem III, similar estimates hold (with $g = 0$ and $A - CKu$ replaced by $A - (d + 2Ku)C$ in (4.52)) when (4.48) is satisfied.

Set $\mathbf{u} = \mathbf{w} + \mathbf{u}^0$. Then $\|\mathbf{u}\|_1 \le \|\mathbf{w}\|_1 + Ku$ and

(i) $a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, div\mathbf{v})_0 - [\lambda, v_n]_k = (\mathbf{f}, \mathbf{v})_0 \;\forall\, \mathbf{v} \in V_k$,

(ii) $(q, div\mathbf{u})_0 + [\tau, u_n]_k = 0 \;\forall\, (q, \tau) \in L^2(\Omega) \times N_k$.

Define $\mathbf{u}^{0'}, (\mathbf{w}', p', \lambda')$, $\mathbf{u}'$ and $(\mathbf{u}^*, p^*, \lambda^*)$ in the same way as was done for the Stokes problem in section 4.2. Then it again follows that $\mathbf{u}^* \in V_k, div\mathbf{u}^* = 0$ and $u_n^* = 0$, so that

$$a_0(\mathbf{u}^*, \mathbf{v}) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{v}) - a_1(\mathbf{u}', \mathbf{u}', \mathbf{v}) = 0 \;\forall\, \mathbf{v} \in V_k.$$

Set $\mathbf{v} = \mathbf{u}^*$. Then we have

$$\begin{aligned} 0 &= a_0(\mathbf{u}^*, \mathbf{u}^*) + a_1(\mathbf{u}, \mathbf{u}, \mathbf{u}^*) - a_1(\mathbf{u}', \mathbf{u}', \mathbf{u}^*) \\ &= a_0(\mathbf{u}^*, \mathbf{u}^*) + a_1(\mathbf{u}^*, \mathbf{u}, \mathbf{u}^*) + a_1(\mathbf{u}', \mathbf{u}^*, \mathbf{u}^*) \\ &\ge (\alpha_0 - \|a_1\|(\|\mathbf{u}\|_1 + \|\mathbf{u}'\|_1))\|\mathbf{u}^*\|_1^2. \end{aligned}$$

Hence, via (4.52) it follows that $\mathbf{u}^* = 0$ when $g/\nu^2 + u$ is sufficiently small. It then follows, by exactly the same argument as for the Stokes problem, that $p^* = 0$ and $\lambda^* = 0$. thus, for a given $\mathbf{u}_0, (\mathbf{u}, p, \lambda)$ is independent of the choice of $\mathbf{u}^0$ if $g/\nu^2 + u$ is sufficiently small.

Furthermore, it is clear from the discussion of section 4.2 that problem (Var) is related to problem (Aux) in precisely the same way as in the case of the Stokes problem, since the nonlinear term can easily be incorporated into the arguments used there.

If the existence and uniqueness results for the problems (Var) associated with problems I - IV are compared to the well-known theory of the corresponding Dirichlet problems, the following is clear:

(1) For the Stokes problem, the situations are essentially identical : for data (external forces, prescribed velocity) of arbitrary magnitude, there exist a corresponding velocity field $\mathbf{u}$, which is uniquely determined, and a pressure field p, which is unique up to an additive constant (with the exception of problem III, where p is unique due to the condition $\tau(\mathbf{u},p)\mathbf{n}\cdot\mathbf{n}=0$ on Λ_o and the uniqueness of $\mathbf{u}$), which constitute (part of) a solution.

(2) In the case of the Navier-Stokes problem, however, there is a fundamental difference. For the Dirichlet problem, solutions $(\mathbf{u},p)$ exist for data of any magnitude. For sufficiently small data, $\mathbf{u}$ is unique and p is unique up to an additive constant. For problems I - IV, existence of solutions can apparently only be established for small data. Moreover, for problem III the solution is possibly non-unique regardless of how small the data is (since the theorem above only shows that $\mathbf{u}$ is unique in some neighbourhood of $\mathbf{0}$).

The reason why the theory for the Dirichlet problem fails to carry over to problem (Var) in the nonlinear case is the following:
In the case of the Dirichlet problem, the form $a(\cdot;\cdot,)$ is exactly as in the present section, but $V = H_0^1(\Omega)^2$ and $B\mathbf{v} = div\mathbf{v}$ so that $K = \{\mathbf{v}\in H_0^1(\Omega)^2 \mid div\mathbf{v}=0\} = K_0$, say. Consequently, the following result can be used to prove that $a(\cdot;\cdot,\cdot)$ satisfies (4.37):

<u>Lemma</u> Let Ω be a bounded domain in $\Re^2$ with a Lipschitz continuous boundary $\partial\Omega$. Let $\Gamma_1,\dots,\Gamma_m$ denote the connected components on $\partial\Omega$. Then, given a function $\mathbf{u}_0 \in H^{\frac{1}{2}}(\partial\Omega)^2$ with $\int_{\Gamma_i}\mathbf{g}\cdot\mathbf{n}\,ds = 0$, $i=1,\dots,m$, there exists for any $\varepsilon>0$ a function

$\mathbf{u}^0 = \mathbf{u}^0(\varepsilon) \,\epsilon\, H^1(\Omega)^2$ such that

$div\mathbf{u}^0 = 0, \;\; \gamma(\mathbf{u}^0) = \mathbf{u}_0, \; | \, a_1(\mathbf{v}, \mathbf{u}^0, \mathbf{v}) \, | \leq \; \varepsilon \, | \, \mathbf{v} \, |_1^2 \;\; \forall \, \mathbf{v} \,\epsilon\, K_0. \, \square$

Together with Korn's inequalities, this lemma represents the main technical step in solving the nonhomogeneous Dirichlet problem. The proof (due to E. Hopf) can be found in Girault and Raviart (1986), pp. 287 - 291, and (for a C^2 domain) in Temam (1979), pp. 175 - 178. Unfortunately, this proof fails work when K_0 is replaced by the spaces K associated with problems I - IV.

We have now completed our study of problem (Aux). It remains to make some remarks on our approach in general. Firstly, the variational formulation (Var) of problem (Aux) is not the only possibility. From its derivation it is clear that all the Dirichlet boundary conditions can be handled via Lagrange multiplies. With respect to those portions of the boundary where mixed conditions are applied (e.g., Γ, Λ_s and Λ_o in problem III), the corresponding multiplier yields a functional representation of one component of the surface stress, a quantity that would otherwise not be available since it does not have a well-defined trace. In view of equation (2.16), the use of the multiplier λ is natural. Observe that the use of additional multipliers does not affect the K-ellipticity of the form a (which is perhaps the most crucial issue) since the space K remains unchanged (although V is replaced by a larger space). However, this introduction of additional variables would complicate the analysis without producing any compensatory advantage. The incompressibility constraint is enforced via a Lagrange multiplier because this formulation arises naturally in the analysis and is a standard approach used in finite element studies of similar problems. Moreover, the multiplier p represents a physically meaningful and important quantity.

Finally, one cannot fail to observe the significant role of arguments based on the solution of fixed point problems in establishing the theory used for our study of the Navier-Stokes problem:

the analysis of the classical Navier-Stokes problem given in Ladyzhenskaya (1969) is based on the Leray-Schauder principle, which is a consequence of the Brouwer fixed point theorem;

the proof of Theorem 4.4 utilizes a consequence of Brouwer's fixed point theorem;

the proof of Theorem 4.5 is based on the contraction principle. Moreover, the basic strategy for solving problem (FBP) itself is usually based on the contraction principle.

5 Free Surface Dependence of the Auxiliary Problem

In this chapter we shall consider step (b) of the strategy set out in section 2.3 for solving problem (FBP). The major part of the chapter is devoted to establishing the continuous dependence of the function λ, which appears in the solution of problem (Var), on the position of the free surface (section 5.1). The remaining aspects of step (b) and the approach as a whole are then discussed briefly (section 5.2).

5.1 Dependence of (Var) on the free surface position

The method used here is based on the same strategy as that of Pukhnachov (1972a) (with the relevant differential equations and Schauder estimates replaced by the corresponding variational equations and estimates given in sections 4.2 and 4.4). Although the procedure will be carried out only for the case of problem IV (with no slip), similar arguments apply in the case of the other problems.

It is clear from the results of the previous section and the results mentioned in chapter 1 that we can only hope to solve problem (FBP) when $D = \|\mathbf{u}_0\|_\Sigma + g/\nu^2$ is in some neighbourhood of zero. Thus, consider the special case when $D = 0$, i.e., the static problem with zero gravity. Then for every f satisfying the relevant conditions, the general solution of problem (Var) is $(\mathbf{u}, p, \lambda) = (\mathbf{0}, r, r\tau^*)$, $r \in \Re$, with τ^* as in (4.20). Via a suitable choice of axes it can be ensured that condition (2.18) is equivalent to

(5.1) $\int_0^1 f(t)dt = 0.$

Thus, to determine the position of the free surface it remains to find a non-negative constant p and a function f_0 which satisfy (2.16), (2.17IV) and (5.1). Set $k = \mu\nu/\sigma L$ and assume that p_a is given as a function of x_1 in $L^1(0,1)$. Then (2.16) becomes

$$\left(f_0' \Big/ \sqrt{1+(f_0')^2}\right)'(x) = k(-p+p_a(x)),\ \ 0 \le x \le 1.$$

If p_a is sufficiently close to a constant (i.e., when $\int_0^1 \mid p_a - \int_0^1 p_a dx \mid dy$ is small enough) the problem can be solved by integrating twice, to obtain

(5.2) $p = \int_0^1 p_a\, dx - 2c/k\sqrt{1+c^2}$,

$$f_0(x) = \int_0^x Q(t)dt - \int_0^1 \int_0^x Q(t)dtdx,\ \ 0 \le x \le 1,$$

where $Q(t) = P(t) \Big/ \sqrt{1-P(t)^2}$ and

$$P(t) = k\left(\int_0^t p_a(s)\, ds - t\int_0^1 p_a(s)\, ds\right) + c(2t-1)/\sqrt{1+c^2},\ \ 0 \le t \le 1.$$

If it is assumed that $p_a \in L^\infty(0,1)$ (essentially bounded), then it can be proved that $f_0 \in C^{1,1}(0,1)$.

In order to minimize the technical detail of the presentation, we shall assume that the domain has the following simple geometry:
For every $f \in \mathcal{F}(f_0,\varepsilon) = \left\{f \in C^{1,1}(0,1) \mid \int_0^1 f dx = 0, -f'(0) = c = f'(1),\right.$ $\left.\|f - f_0\|_{1,1} \le \varepsilon\right\}$, set $\Omega(f) = \{\mathbf{x} \in \Re^2 \mid 0 < x_1 < 1,\ -h < x_2 < f(x_1)\}$, $\Sigma = \{\mathbf{x} \in \Re^2 \mid 0 < x_1 < 1,\ x_2 = -h\}$ and define $\Gamma(f), \Lambda_0$ and Λ_1 as before. (By taking p_a close to a constant and c small enough, it can be ensured that $\|f_0\|_{1,1} \le \delta/2$, say. Then $\|f\|_{1,1} \le \delta$ when $\varepsilon \le \delta/2$.) Our aim it to show that the mapping $f \to \lambda(f)$ is Lipschitz continuous on $\mathcal{F}(f_0,\varepsilon)$ (where λ is defined as in Theorem 4.3 when $e = 0$ and as in Theorem 4.7 when $e = 1$.) For this it is necessary to define a norm on $N(f) = H_{00}^{\frac{1}{2}}(\Gamma(f))$ which is independent of f.

If $v \in H_{00}^{\frac{1}{2}}(\Gamma(f))$, then $v(\mathbf{x}) = v(x_1)$. Hence, the inner product defined in (3.7) can be written as $(\mathbf{u},\mathbf{v})_{00} = I_1 + I_2 + I_2$ where

$$I_1 = \int_{\Gamma(f)} uv\, ds = \int_0^1 u(x)v(x)S(f)(x)\, dx \text{ with } S(f)(x) = \sqrt{1+f'(x)^2}$$

$$I_2 = \int_{\Gamma(f)} uv/\rho\, ds, \text{ etc., and}$$

$$
\begin{aligned}
I_3 &= \int_{\Gamma(f)} \int_{\Gamma(f)} R(u,v)(x,y)/d(x,y)\; ds(x)\, ds(y) \\
&= \int_0^1 \int_0^1 R(u,v)(x,y) S(f)(x) S(f)(y)/d(x,y)\; dx\, dy
\end{aligned}
$$

where $R(u,v)(x,y) = (u(x)-u(y))(v(x)-v(y))$, $d(x,y) = |\, x-y \,|^2 + |\, f(x)-f(y) \,|^2$ and $s(x) = \int_0^x S(f)(t)\, dt$.

Since $1 \le S(f)(x) \le \sqrt{1+\delta^2}$ and $|\, x-y \,|^2 \le d(x,y) \le (1+\delta^2)\,|\, x-y \,|^2$, the inner product on $H_{00}^{\frac{1}{2}}(\Gamma(f))$ defined by

(5.3) $(u,v)^{00} = \int_0^1 \int_0^1 R(u,v)(x,y)/\,|\, x-y \,|^2 \; dx\, dy \; + \int_0^1 (1+1/\rho(x)) u(x) v(x)\; dx$

is independent of f and generates a norm $\|\cdot\|^{00}$ which is equivalent to $\|\cdot\|_{00}$. We shall denote the corresponding space by $H_{00}^{\frac{1}{2}}(0,1)$ or N.

Furthermore, to ensure that the representation of $\gamma(S_{ij})n_j n_i$ depends on f only through the dependence of Ω on f, we replace $(\cdot,\cdot)_{00,\Gamma(f)}$ by $(\cdot,\cdot)^{00}$. Then, by arguments identical to those used in chapter 3 to derive problem (Var), we obtain the following weak formulation of problem (Aux):

(W) Given $\mathbf{u}_0 \in H_{00}^{\frac{1}{2}}(\Sigma)^2$ with $\int_\Sigma \mathbf{u}_0 \cdot \mathbf{n}\; ds = 0$, find $(\mathbf{u}, r, \eta) \in V(f) \times Q(f) \times N$ such that

(5.4) $a_0(\mathbf{u},\mathbf{v}) + e a_1(\mathbf{u},\mathbf{u},\mathbf{v}) - (r, div\mathbf{v})_0 - (\eta, v_n)^{00} = (\mathbf{f},\mathbf{v})_0 \;\; \forall \; \mathbf{v} \in V(f)$,

(5.5) $(q, div\mathbf{v})_0 + (\tau, u_n)^{00} \;\; \forall\, (q,\tau) \in M(f)$,

where $V(f) = \{\mathbf{v} \in H^1(\Omega(f))^2 \mid \gamma(\mathbf{v}) = \mathbf{0} \text{ on } \partial\Omega \setminus \Gamma\}, Q(f) = L^2(\Omega(f))$ and $M(f) = Q(f) \times N$.

Since the operator B associated with problem (W) is identical to that of problem (Var), it follows precisely as in chapter 4 that there exist unique functions $\mathbf{u} \in V$ and $(r,\eta) \in S$, with S as in (4.19), such that $(\mathbf{u}, r+t, \eta+t\eta^*)$ is a solution of problem (W) for every $t \in \Re$ (under the assumption that D is sufficiently small when $e = 1$ to ensure that problem (NVP) has a unique solution). Here the function $\eta^*(f) \in N$ is determined uniquely (via the Riesz representation theorem) by the relation

(5.6) $(\eta^*(f), \tau)^{00} = -\int_{\Gamma(f)} \tau \, ds \;\; \forall \, \tau \in N.$

(Since $N(f)$ and N have equivalent norms and are equal as sets, it follows that the identity map $i(f) : N(f)' \to N'$ is an isomorphism. Let $j(f)$ and j denote the respective Riesz maps. Then $\phi(f) = j^{-1} \cdot i(f) \cdot j(f)$ is a (normed space) isomorphism from $N(f)$ onto N with the property that
$(\phi(\lambda), \tau)^{00} = (\lambda, \tau)_{00} \;\; \forall \, \lambda, \; \tau \in N(f) = N.$

By using ϕ and arguments similar to those following (4.52) it can be proved that if $(\mathbf{u}', p, \lambda)$ is the solution with $(p, \lambda) \in S$ (and $\mathbf{u}' = \mathbf{w}(\mathbf{u}^0) + \mathbf{u}^0$) of the corresponding problem (Var), then $\mathbf{u} = \mathbf{u}'$ and (r, η) is the orthogonal projection in $M(f)$ of $(p, \phi(\lambda))$ on S, i.e.,

$(r, \eta) = (p, \phi(\lambda)) - z(1, \eta^*)$ with $z = (\int_\Omega p \, dx - \int_\Gamma \phi(\lambda) ds)/\|(1, \eta^*)\|_M^2)$.

For $m = 1, 2$, let $f_m \in \mathcal{F}(f_0, \varepsilon)$ be fixed, denote $\Omega(f_m)$ by Ω_m, $V(f_m)$ by V_m, etc. and let (u_m, r_m, η_m) be the solution with $(r_m, \eta_m) \in S_m$ of problem $(W)_m$. As in chapter 4, it follows that (when D is sufficiently small) there exists a constant U_m which depends only on Ω_m such that $\|u_m\|_1 + \|r_m\|_0 + \|\eta_m\|^{00} \leq U_m D$. Using this fact, we shall now show that there exists a constant $U = U(h, \delta)$ such that $\|\eta_1 - \eta_2\|^{00} \leq UD\|f_1 - f_2\|$, where $\|\cdot\|$ denotes $\|\cdot\|_{C^{1,1}[0,1]}$.

The first step is to define a smooth (class C^1) bijective mapping $T : \Omega_2 \to \Omega_1 : \mathbf{x} \to \mathbf{z}$. Let g be any strictly monotone increasing (and thus invertible), smooth, real-valued function defined on $[-h, \delta]$. For every $\mathbf{x} \in \Omega_2$, define $\mathbf{z} \in \Omega_1$ by

$$(5.7) \quad \begin{cases} z_1 &= x_1 \\ z_2 &= (f_1(x_1)[g(x_2) - g(-h)] + h[g(x_2) - g(f_2(x_1))])/(g(f_2(x_1)) - g(-h)). \end{cases}$$

It is easy to see that Σ is mapped onto Σ, Γ_2 onto Γ_1, etc. With $g(x) = x$, we obtain the map

$$(5.8)\quad \begin{cases} z_1 &= x_1 \\ z_2 &= (f_1(x_1)(x_2+h) + h(x_2 - f_2(x_1))/(f_2(x_1)+h) \end{cases}$$

with derivatives

$$(5.9)\quad \begin{cases} \partial z_1/\partial x_1 &= 1, \;\; \partial z_1/\partial x_2 = 0 \\ \partial z_2/\partial x_1 &= (z_2+h)\left((f_2+h)f_1' - (f_1+h)f_2'\right)/(f_2+h)(f_1+h) = A(\mathbf{z}), \\ \partial z_2/\partial x_2 &= (f_1(z_1)+h)/(f_2(z_1)+h) = B(\mathbf{z}), \text{ say.} \end{cases}$$

The inverse map is of identical form, with the roles of $\mathbf{x}$ and f_2 exchanged with those of $\mathbf{z}$ and f_1 in (5.8). Furthermore, $\partial(x_1,x_2)/\partial(z_1,z_2) = 1/B = J(\mathbf{z})$, say. Let g be a function defined on Ω_2. Then we denote by $\bar{g}$ the function defined on Ω_1 by $\bar{g}(\mathbf{z}) = g(\mathbf{x})$. If follows that

$$(5.10)\quad \textstyle\int_{\Omega_2} g\, dx \;=\; \int_{\Omega_1} \bar{g} J dz,$$

$$(5.11)\quad g_{,1}(\mathbf{x}) = (\bar{g}_{,1} + A\bar{g}_{,2})(\mathbf{z}),\;\; g_{,2}(\mathbf{x}) = (B\bar{g}_{,2})(\mathbf{z}).$$

It follows from the inequality
$(h-\delta)/(h+\delta) \;\le\; (h+f_0-\varepsilon)/(h+f_0+\varepsilon) \;\le\; J \;\le\; (h+f_0+\varepsilon)/(h+f_0-\varepsilon)$
$\le\; (h+\delta)/(h-\delta)$
that $g \in L^2(\Omega_2)$ iff $\bar{g} \in L^2(\Omega_1)$ and that there exist constants K_1 and K_2, independent of $g, \bar{g}, f_1$ or f_2, such that $K_1\|g\|_0 \;\le\; \|\bar{g}\|_0 \;\le\; K_2\|g\|_0$.

Similarly, for any vector-valued function $\mathbf{v}$ on Ω_2, we define $\bar{\mathbf{v}}$ on Ω_1 by $\bar{\mathbf{v}}(\mathbf{z}) = \mathbf{v}(\mathbf{x})$. Then $(\bar{\mathbf{v}})_i = \overline{v_i}$, $i = 1,2$, so that

$$\|\mathbf{v}\|_1^2 \;=\; \textstyle\int_{\Omega_2}(v_iv_i + v_{i,j}v_{i,j})\, dx \;=\; \int_{\Omega_1}(\bar{v}_i\bar{v}_i + \sum_{i=1}^{2}((\bar{v}_{i,1} + A\bar{v}_{i,2})^2 + (B\bar{v}_{i,2})^2))J\, dz.$$

From the inequality (for arbitrary $a, b \in \Re$)

$-A^{4/3}a^2 - A^{2/3}b^2 \leq 2(A^{2/3}a)(A^{1/3}b) = 2a(Ab) \leq a^2 + A^2b^2$

we get the identity

$(1 - A^{4/3})a^2 + (A^2 - A^{2/3} + B^2)b^2 \leq (a + Ab)^2 + (Bb)^2 \leq 2a^2 + (2A^2 + B^2)b^2.$

Furthermore, it follows from the inequality for J above and the analogous ones for A and B that $J \to 1, A \to 0$ and $B \to 1$ when $\varepsilon \to 0$. Hence, for sufficiently small ε, there exist constants K_3 and K_4, of the type above, such that $K_3\|\mathbf{u}\|_1 \leq \|\bar{\mathbf{u}}\|_1 \leq K_4\|\mathbf{u}\|_1$. Therefore, $\mathbf{u} \in H^1(\Omega_2)^2$ iff $\bar{\mathbf{u}} \in H^1(\Omega_1)^2$.

Finally, for any function v on Γ_2, we define $\bar{v}$ on Γ_1 by $\bar{v}(z_1) = v(x_1)$. Clearly, $(u, v)^{00} = (\bar{u}, \bar{v})^{00} \ \forall\, u, v \in N$.

The next step is to transform the equations (5.4) and (5.5) corresponding to Ω_2 to Ω_1. Since T_2 maps $\partial\Omega_2 \setminus \Gamma_2$ onto $\partial\Omega_1 \setminus \Gamma_1$ and $H^1(\Omega_2)^2$ is mapped onto $H^1(\Omega_1)^2$, it follows that V_2 is mapped onto V_1 by the mapping $\mathbf{v} \to \bar{\mathbf{v}}$. We know already that M_2 is mapped onto M_1. Furthermore, for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_2$ and $(q, \tau) \in M_2$,

$$
\begin{aligned}
a_0(f_2)(\mathbf{u}, \mathbf{v}) &= 2\int_{\Omega_2} D_{ij}(\mathbf{u})D_{ij}(\mathbf{v})\, dx \\
&= \int_{\Omega_2} (2u_{1,1}v_{1,1} + (u_{1,2} + u_{2,1})(v_{1,2} + v_{2,1}) + 2u_{2,2}v_{2,2})\, dx \\
&= \int_{\Omega_1} J[2(\bar{u}_{1,1} + A\bar{u}_{1,2})(\bar{v}_{1,1} + A\bar{v}_{1,2}) + 2(B\bar{u}_{2,2})(B\bar{v}_{2,2}) \\
&+ (B\bar{u}_{1,2} + \bar{u}_{2,1} + A\bar{u}_{2,2})(B\bar{v}_{1,2} + \bar{v}_{2,1} + A\bar{v}_{2,2})]\, dz \\
&= \bar{a}_0(\bar{\mathbf{u}}, \bar{\mathbf{v}}); \\
a_1(f_2)(\mathbf{w}, \mathbf{u}, \mathbf{v}) &= \int_{\Omega_2} w_j u_{i,j} v_i\, dx \\
&= \int_{\Omega_1} J[\bar{w}_1(\bar{u}_{1,1} + A\bar{u}_{1,2})\bar{v}_1 + \bar{w}_2(B\bar{u}_{1,2})\bar{v}_1 + \bar{w}_1(\bar{u}_{2,1} + A\bar{u}_{2,2})\bar{v}_2 \\
&\quad + \bar{w}_2(B\bar{u}_{2,2})\bar{v}_2]\, dz \\
&= \bar{a}_1(\bar{\mathbf{w}}, \bar{\mathbf{u}}, \bar{\mathbf{v}});
\end{aligned}
$$

$(q, div\mathbf{v})_{0,\Omega_2} = \int_{\Omega_2} qv_{i,i}\, dx = \int_{\Omega_1} J\bar{q}d(\bar{\mathbf{v}})\, dz = (\bar{q}, Jd(\bar{\mathbf{v}}))_{0,\Omega_1}$

where $d(\bar{\mathbf{v}}) = \bar{v}_{1,1} + A\bar{v}_{1,2} + B\bar{v}_{2,2}$;

$(\tau, \gamma_{\Gamma_2} n(\mathbf{v}))^{00} = (\bar{\tau}, n(\bar{\mathbf{v}}))^{00}$ where $n(\bar{\mathbf{v}}) = \overline{v_n} = (\gamma(\bar{v}_2) - \gamma(\bar{v}_1) f_2') \Big/ \sqrt{1 + (f_2')^2}$;

$(\mathbf{f}, \mathbf{v})_{0,\Omega_2} = (\mathbf{f}_2, \bar{\mathbf{v}})_{0,\Omega_1}$ where $\mathbf{f}_2 = J\mathbf{f}$ with $\mathbf{f}$ as in (4.16).

With these definitions, equations (5.4) and (5.5) now become

$$(5.12)\ \ \bar{a}_0(\overline{\mathbf{u}_2}, \mathbf{v}) + e\bar{a}_1(\overline{\mathbf{u}_2}, \overline{\mathbf{u}_2}, \mathbf{v}) - (\overline{r_2}, Jd(\mathbf{v}))_0 - (\overline{\eta_2}, n(\mathbf{v}))^{00} = (\mathbf{f}_2, \mathbf{v})_0 \ \ \forall\, \mathbf{v} \in V_1,$$

$$(5.13)\ \ (q, Jd(\overline{\mathbf{u}_2}))_0 + (\tau, n(\overline{\mathbf{u}_2}))^{00} = 0 \ \ \forall\, (q, \tau) \in M_1.$$

Set $\mathbf{u} = \mathbf{u}_1 - \overline{\mathbf{u}_2}$, $r = r_1 - \overline{r_2}$ and $\eta = \eta_1 - \overline{\eta_2}$. Then, by subtracting equations (5.12) and (5.13) from (5.4) and (5.5) for Ω_1 and rearranging terms, we obtain:

$$(5.14)\ \ a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, (r, \eta)) = < l, \mathbf{v} > \ \ \forall\, \mathbf{v} \in V_1,$$

$$(5.15)\ \ \qquad b(\mathbf{u}, (q, \tau)) = < k, (q, \tau) > \ \ \forall\, (q, \tau) \in M_1,$$

where $a(\mathbf{u}, \mathbf{v}) = a_0(\mathbf{u}, \mathbf{v}) + ea_1(\mathbf{u}_1, \mathbf{u}, \mathbf{v}) + ea_1(\mathbf{u}, \overline{\mathbf{u}_2}, \mathbf{v})$,

$b(\mathbf{v}, (q, \tau)) = -(q, div\mathbf{v})_0 - (\tau, v_n))^{00}$,

$< l, \mathbf{v} > = (\mathbf{f} - \mathbf{f}_2, \mathbf{v})_0 + \bar{a}_0(\overline{\mathbf{u}_2}, \mathbf{v}) - a_0(\overline{\mathbf{u}_2}, \mathbf{v}) + e\bar{a}_1(\overline{\mathbf{u}_2}, \overline{\mathbf{u}_2}, \mathbf{v}) - ea_1(\overline{\mathbf{u}_2}, \overline{\mathbf{u}_2}, \mathbf{v})$
$+ (\overline{r_2}, div\mathbf{v} - Jd(\mathbf{v}))_0 + (\overline{\eta_2}, v_n - n(\mathbf{v}))^{00}$,

$< k, (q, \tau) > = (q, div(\overline{\mathbf{u}_2}) - Jd(\overline{\mathbf{u}_2}))_0 + (r, (\overline{\mathbf{u}_2})_n - n(\overline{\mathbf{u}_2}))^{00}$.

We shall refer to equations (5.14) and (5.15) as problem (D). From the inequalities derived earlier and the fact that they are linear and bijective, it follows that the mappings $\mathbf{v} \to \bar{\mathbf{v}} : V_2 \to V_1$, $q \to \bar{q} : Q_2 \to Q_1$ and $\tau \to \bar{\tau} : N \to N$ are isomorphisms. For simplicity, we shall denote these maps and their inverses jointly by F and R, respectively. The precise meaning will be clear form the context in which the symbol is used.

By using the properties of F, R and those established in sections 4.2 and 4.4 for the forms $a_0(f_m)(\cdot,\cdot)$, $a_1(f_m)(\cdot,\cdot,\cdot)$ and $b(f_m)(\cdot,\cdot)$, it is easy to prove that the forms $a(\cdot,\cdot)$ and

$b(\cdot,\cdot)$ defined above are continuous bilinear forms on respectively and that $V_1 \times V_1$ and $V_1 \times M_1$, respectively, and that $l \in V_1'$ and $k \in M_1'$. For example, the boundedness of $a(\cdot,\cdot)$ is proved as follows:

$$| a(\mathbf{w},\mathbf{v}) | = | a_0(f_1)(\mathbf{w},\mathbf{v}) + ea_1(f_1)(\mathbf{u}_1,\mathbf{w},\mathbf{v}) + ea_1(f_2)(R\mathbf{w},\mathbf{u}_2,R\mathbf{v}) |$$

$$\leq (\|a_0(f_1)\| + e\|a_1(f_1)\| \cdot \|\mathbf{u}_1\|_1)\|\mathbf{w}\|_1\|\mathbf{v}\|_1 + e\|a_1(f_2)\| \cdot \|\mathbf{u}_2\|_1\|R\mathbf{w}\|_1\|R\mathbf{v}\|_1$$

$$\leq (4 + eC(\Omega_1)U_1D + eC(\Omega_2)U_2D/K_3^2)\|\mathbf{w}\|_1\|\mathbf{v}\|_1 \ \forall\, \mathbf{w},\mathbf{v} \in V_1.$$

It is clear that the operator B associated with problem (D) is identical to the operator B_1 of problem $(W)_1$, so that $Rg(B) = S_1$, etc. Furthermore, for all $\mathbf{v} \in K = K_1 = \{\mathbf{v} \in V_1 \mid div\mathbf{v} = 0, v_n = 0\}$,

$$\begin{aligned}
a(\mathbf{v},\mathbf{v}) &= a_0(\mathbf{v},\mathbf{v}) + ea_1(\mathbf{u}_1,\mathbf{v},\mathbf{v}) + ea_1(f_2)(R\mathbf{v},\mathbf{u}_2,R\mathbf{v}) \\
&\geq \alpha_0\|\mathbf{v}\|_1^2 - e\|a_1(\Omega_2)\| \cdot \|\mathbf{u}_2\|_1\|R\mathbf{v}\|_1^2 \\
&\geq (A(\Omega_1) - eC(\Omega_2)U_2D/K_3^2)\|\mathbf{v}\|_1^2 \\
&\geq \alpha\|\mathbf{v}\|_1^2
\end{aligned}$$

where $\alpha = \alpha(\Omega_1) > 0$ when $e = 0$, and $\alpha = \alpha(\Omega_1,\Omega_2,D) > 0$ if D is sufficiently small when $e = 1$. (Using the properties of R, a_0 and a_1, the estimate for $\|\mathbf{u}_1\|_1$ derived in section 4.4, and the proof (4.44).)

Hence, from section 4.1 it follows that there exists a unique solution $(\mathbf{u}',(r',\eta'))$ in $V_1 \times S_1$ of problem (D). Moreover, by (4.14),

$$(5.16)\quad \|(r',\eta')\|_{M_1} \leq (1 + \|a\|/\alpha)(\|l\|_{V_1'} + \|a\| \cdot \|k\|_{M_1'}/\beta)/\beta$$

where $\beta(\Omega_1) = \inf\limits_{(q,\tau)\in S_1} \sup\limits_{\mathbf{v}\in V_1} b(\mathbf{v},(q,\tau))/\|\mathbf{v}\|_1\|(q,\tau)\| > 0$ (by the proof of Theorem 4.2(b)). (Note that it is not necessary to know whether k satisfies the condition stated in Theorem 4.2(b) since (5.16) holds in both cases.) Since $(\mathbf{u},(r,\eta)) \in V_1 \times M_1$ is also a solution of problem (D), it follows that $\mathbf{u}' = \mathbf{u}$ and that (r',η') is the orthogonal projection in M_1 of (r,η) on S_1, i.e.,

$(r', \eta') = (r, \eta) - (y/z^2)(1, \eta_1^*)$ with $y = ((r, \eta), (1, \eta_1^*))_{M_1} = -((\overline{r_2}, \overline{\eta_2}), (1, \eta_1^*))_{M_1}$ (since $(r_1, \eta_1) \in S_1$) and $z = \|(1, \eta_1^*)\|_{M_1}$. This implies that

(5.17) $\|\eta_1 - \eta_2\|^{00} \leq \|(r, \eta)\|_{M_1} \leq \|(r', \eta')\|_{M_1} + |\, y \,| / z.$

We shall now show that each of the two terms in the right hand side is bounded from above by a term of the form $C\|f_1 - f_2\|_1$ with the constant C independent of f_1 and f_2.

Since $(r_2, \eta_2) \in Rg(B_2) = S_2 = \{(q, \tau) \in M_2 \mid ((q, \tau), (1, \eta_2^*))_{M_2} = 0\}, (\overline{r_2}, \overline{\eta_2}) \in F(S_2) = \left\{(p, \lambda) \in M_1 \mid ((p, \lambda), (J, \overline{\eta_2^*}))_{M_1} = 0\right\}$ and thus

$$
\begin{aligned}
|\, y \,| &\leq |\, ((\overline{r_2}, \overline{\eta_2}), (J, \overline{\eta_2^*}))_{M_1} \,| + |\, ((\overline{r_2}, \overline{\eta_2}), (1, \eta_1^*) - (J, \overline{\eta_2^*}))_{M_1} \,| \\
&\leq t\|(\overline{r_2}, \overline{\eta_2})\|_{M_1}
\end{aligned}
$$

where $t = \|(1 - J, \eta_1^* - \overline{\eta_2^*})\|_{M_1} \leq u + v$

with $u = \|1 - J\|_0 = \|(f_1 - f_2)/(f_1 + h)\|_0 \leq \sqrt{Vol}\|f_1 - f_2\|/(h - \delta)$

$$
\begin{aligned}
\text{and } v &= \|\eta_1^* - \overline{\eta_2^*}\|^{00} \\
&= \sup_{\tau \,\in\, N} \int_0^1 \tau(z)(S(f_1)(z) - S(f_2)(z))\, dz / \|\tau\|^{00} \\
&\leq \left(\sup_{[0,1]} |\, S(f_1) - S(f_2) \,|\right) \left(\sup_{\tau \,\in\, N} \|\tau\|_0 / \|\tau\|^{00}\right) \\
&\leq \sup_{[0,1]} |\, f_1' - f_2' \,| \cdot |\, f_1' + f_2' \,| / 2 \\
&\leq \delta\|f_1 - f_2\|.
\end{aligned}
$$

(Using the definitions of $\eta_i^*, i = 1, 2$, and the fact that the Riesz map is an isometry.)

Since $z \geq \|1\|_0 = \sqrt{Vol}$, it follows that

(5.18) $|\, y \,| / z \leq U_3 D\|f_1 - f_2\|$ with $U_3 = K_4 U_2 (1/(h - \delta) + \delta/\sqrt{Vol})$.

(2) For clarity, let $\overline{\mathbf{u_2}}$ be denoted by $\mathbf{w}$. For every $(q, \tau) \in M_1, |< k, (q, \tau) >| \leq k_1 + k_2$ with

$$k_1 = \|q\|_0\|div\mathbf{w} - Jd(\mathbf{w})\|_0 = \|q\|_0\|(1-J)w_{1,1} + (1-B)w_{2,2} - Aw_{1,2}\|_0$$
$$\leq \|q\|_0(k_3+k_4)\|\mathbf{w}\|_1 \leq U_2D(k_3+k_4+k_5)\|q\|_0$$

where $k_3 = \sup_{[0,1]} |1-J| = \sup_{[0,1]} |(f_1-f_2)/(f_1+h)| \leq \|f_1-f_2\|/(h-\delta)$,

$k_4 = \sup_{[0,1]} |1-B| = \sup_{[0,1]} |f_2-f_1|/|f_2+h| \leq \|f_1-f_2\|/(h-\delta)$

and $k_5 = \sup_{[0,1]} |A| \leq ((h+\delta)/(h-\delta)).\sup_{[0,1]}(|f_2+h|\cdot|f_1'-f_2'| + |f_2-f_1|\cdot|f_2'|)$
$\leq (h+\delta)(h+2\delta)\|f_1-f_2\|/(h-\delta)^2$;

$k_2 = \|\tau\|^{00}\|w_n - n(\mathbf{w})\|^{00} \leq (k_6+k_7)\|\tau\|^{00}$ where

$k_6 = \|(1/S(f_1) - 1/S(f_2))\gamma(w_2)\|^{00}$ and $k_7 = \|(f_1'/S(f_1) - f_2'/S(f_2))\gamma(w_1)\|^{00}$
(with $S(f)$ defined as in the paragraph preceding (5.3)).

In general, if $v \in C^{0,1}[0,1]$ and $w \in N$, then $vw \in N$ and $\|vw\|^{00} \leq \sqrt{3}\|v\|_{0,1}\|w\|^{00}$.
(This follows from the definitions of $N, \|\cdot\|^{00}$ and the following inequalities:

$$(vw)(x)^2\cdot(1+1/\rho)(x) \leq (\sup_{[0,1]}|v|)^2 w(x)^2(1+1/\rho)(x) \quad \forall x \in [0,1],$$

$$\begin{aligned}
&[(vw)(x) - (vw)(y)]^2/(x-y)^2\\
&= [(v(x)-v(y))w(x) + v(y)(w(x)-w(y))]^2/(x-y)^2\\
&\leq 2[(v(x)-v(y))/(x-y)]^2 w(x)^2 + 2v(y)^2[(w(x)-w(y))/(x-y)]^2\\
&\leq \|v\|_{0,1}^2\cdot w(x)^2 + 2(\sup_{[0,1]}|v|)^2[(w(x)-w(y))/(x-y)]^2 \;\forall x,y \in [0,1], x\neq y.)
\end{aligned}$$

Choose $v = 1/S(f_1) - 1/S(f_2)$ and $w = \gamma(w_2)$. Then $w \in N$ with $\|w\|^{00} \leq tU_2D/K_3$ where $t(\Omega_2) = \|\gamma_{\Gamma_1}\|$, so that it remains to show that $v \in C^{0,1}[0,1]$ with $\|v\|_{0,1} \leq C(h,\delta)\|f_1-f_2\|$. It is clear that v is continuous on $[0,1]$. Let f_1 and f_2 be denoted by respectively m and n. Then, for all $x,y \in [0,1], x\neq y$,

$$\begin{aligned}
|v(x)| &= |1/(1+m(x)^2) - 1/(1+n(x)^2)| / |1/\sqrt{1+m(x)^2} + 1/\sqrt{1+n(x)^2}|\\
&\leq |n(x)-m(x)|\cdot|m(x)+n(x)| / ((2/\sqrt{1+\delta^2})(1+m(x)^2)(1+n(x)^2))
\end{aligned}$$

$$\leq \ \delta\sqrt{1+\delta^2}\|f_1 - f_2\|,$$

$$\begin{aligned} | v(x) \ - \ v(y) | &= | (n-m)(x)E(x) - (n-m)(y)E(y) | \\ &\leq | (n-m)(x) - (n-m)(y) | \cdot | E(x) | + | (n-m)(y) | \cdot | E(x) - E(y) | \\ &\leq \|E\|_{0,1}\|f_1 - f_2\| \cdot | x - y | \end{aligned}$$

where the function $E \in C^{0,1}[0,1]$ is defined by $E = (m+n)/((1+m^2)(1+n^2)(1/\sqrt{1+m^2} + 1/\sqrt{1+n^2}))$. ($E$ is bounded since $| E(x) | \leq (\delta+\delta)/((1 \cdot 1 \cdot (1/\sqrt{1+\delta^2}))$. The Lipschitz continuity of E follows from the fact that $m, n \in C^{0,1}[0,1]$ and repeated applications of the identity $| F(x)G(x) - F(y)G(y) | \leq | F(x) \cdot F(y) | \cdot | G(x) | + | F(y) | \cdot | G(x) - G(y) |$). It is easy to see that there exist a constant $k_8(h,\delta)$ such that $\|E\|_{0,1} \ \leq \ k_8$. Hence,
$k_6 \leq \sqrt{3}(tU_2/K_3)(k_8 + \delta\sqrt{1+\delta^2})D\|f_1 - f_2\|$.
A similar bound holds for k_7. This proves that there exists a constant $U_4 = U_4(t, U_2, h, \delta)$ such that

(5.19) $\|k\|_{M_1'} \ \leq U_4 D\|f_1 - f_2\|$.

It follows from similar arguments that there exists a constant U_5, of the same type as U_4, such that

(5.20) $\|l\|_{V_1'} \ \leq U_5 D\|f_1 - f_2\|$.

(3) The only remaining question is whether the coefficient of $\|f_1 - f_2\|$ in each of the inequalities above can be chosen independently of f_1 and f_2. This will be the case if the following conditions are satisfied:

(A) There exist positive constant α_0^*, β^*, t^* and C^*, independent of f, such that for every $f \in \mathcal{F}(f_0, \varepsilon)$,

(5.21) $\alpha(f) = \inf_{\mathbf{v} \in V(f)} a_0(f)(\mathbf{v},\mathbf{v})/\|\mathbf{v}\|_1^2 \ \geq \ \alpha_0^*,$

(5.22) $\beta(f) = \inf_{(q,\tau)\,\epsilon\, S} \sup_{v \epsilon V(f)} b(f)(v,(q,\tau))/\|v\|_1\|(q,\tau)\| \geq \beta^*,$

(5.23) $t(f) = \|\gamma_{\Gamma(f)}\|_{\mathcal{L}(V(f),N)} \leq t^*,$

(5.24) $C(f) = \|a_1(f)\|_{(V(f)^3)'} \leq C^*,$

or equivalently, $\inf_{f \,\epsilon\, \mathcal{F}(f_0,\varepsilon)} min(\alpha(f),\beta(f)) > 0, \quad \sup_{f \,\epsilon\, \mathcal{F}(f_0,\varepsilon)} max(C(f),t(f)) < +\infty.$

To prove (5.23), let f_0 be the function defined in (5.2) and fix an arbitrary $f \,\epsilon\, \mathcal{F}(f_0,\varepsilon)$. Now define the mappings $x \to \mathbf{z} : \Omega(f_0)$ and $\mathbf{v} \to \bar{\mathbf{v}} : H^1(\Omega)(f))^2 \to H^1(\Omega(f_0))^2$ in the same way as before (with f_1, f_2 replaced by f_0, f_1). Then $\|\gamma_{\Gamma(f)}\mathbf{v}\|_N = \|\gamma_{\Gamma(f_0)}\bar{\mathbf{v}}\|_N$ and $\|\bar{\mathbf{v}}\|_1 \leq K_4(h,\delta)\|\mathbf{v}\|_1 \;\; \forall\, \mathbf{v} \,\epsilon\, V(f)$, so that $t(f) = \sup_{\mathbf{v}}\|\gamma_{\Gamma(f)}\mathbf{v}\|_N/\|\mathbf{v}\|_1 \leq$ $\sup_{\mathbf{v}} K_4\|\gamma_{\Gamma(f_0)}\bar{\mathbf{v}}\|_N/\|\bar{\mathbf{v}}\|_1 \leq K_4 t(f_0) = t^*.$

Condition (5.24) follows directly from the proof of Lemma 4.5(a) because the constant c associated with the imbedding of $H^1(\Omega(f))$ into $L^4(\Omega(f))$ depends at most on $meas(\Omega(f))$, which is independent of f due to (5.1) (cf. Adams (1975), 2.8, 5.13, 5.14).

It is not obvious how to prove (5.21) or (5.22) since the posivity of the constants $\alpha(f)$ and $\beta(f)$ was established via nonconstructive methods. However, a plausible conjecture is that the mappings $f \to \alpha(f)$ and $f \to \beta(f)$ from $\mathcal{F}(f_0,\varepsilon)$ into $\Re$ are continuous in some neighbourhood of f_0, i.e., for some small value of ε. Under this assumption there exists an ε such that (5.21) and (5.22) hold with, say, $\alpha_0^* = \alpha_0(f_0)/2$ and $\beta^* = \beta(f_0)/2$.

Furthermore, under the assumption above it follows that when $e = 0$, $\|a\|$ can be bounded from above and α from below by constants which are independent of f_1, f_2 and D. When $e = 1$ there exists a constant D^*, independent of f_1 and f_2, such that $\|a\|$ and α can be bounded in the same was as when $e = 0$ whenever $D \leq D^*$. Since U_4 and U_5 are now also independent of f_1 and f_2, it follows from (5.16) that there exists a constant $U_6 = U_6(h,\delta)$ such that

(5.25) $\|(r',\eta')\|_{M_1} \leq U_6 D\|f_1 - f_2\|.$

Hence, by virtue of (5.17), (5.18) and (5.25) we may set $U = U_3 + U_6$. In summary, we have the following result:

Theorem 5.1 Under the hypothesis that the inequalities (5.21) and (5.22) hold (with, say, $\varepsilon = \varepsilon^*$), it follows that

(a) in the case of the Stokes problem ($e = 0$), there exist constants $R_1(h), R_2(h, \delta, \varepsilon^*)$ and $U(h, \delta)$ such that if $0 < \delta \leq R_1$ and $0 < \varepsilon \leq R_2$, then

(5.26) $\|\eta(f_1) - \eta(f_2)\|^{\infty} \leq UD\|f_1 - f_2\| \;\; \forall f_1, f_2 \in \mathcal{F}(f_0, \varepsilon)$;

(b) in the case of the Navier-Stokes problem ($e = 1$), there exist constants $R_3(h)$, $R_4(h, \delta, \varepsilon^*)$,
$R_5(h, \delta)$ and $U(h, \delta)$ such that (5.26) holds if $0 < \delta \leq R_3, 0 < \varepsilon \leq R_4$ and $0 \leq U \leq R_5$. □

5.2 The regularity problem

In chapter 4 it was proved under minimal conditions on the smoothness of the data (namely that $\partial\Omega$ is Lipschitz continuous, f is of class $C^{1,\alpha}$ with $\alpha > \frac{1}{2}, \mathbf{f} \in L^2(\Omega)^2, p_a \in \mathcal{L}^{\infty}(\Gamma)$, and $\mathbf{u}_0$ belongs to (some subspace of) $H^{\frac{1}{2}}(\Sigma)^2$) that for every choice of the function f there exists a pair of functions $(\mathbf{u}(f), p(f))$ in $H^1(\Omega)^2 \times L^2(\Omega)$ which is a weak solution of problem (Aux). In order to derive (5.26) we assumed furthermore that $f \in C^{1,1}[0,1]$. This implies that f'' is defined a.e. in (0, 1) and belongs to the space $\mathcal{L}^{\infty}(0,1)$ of essentially bounded Lebesgue-measurable functions. In fact,
$\|f''\|_{\infty} = inf\,\{b \in \Re \mid | f''(x) | \leq b \text{ for a.e. } x \text{ in } (0,1)\} \leq \|f\|_{1,1}$.

It follows that the curvature operator $(1/R)(f)$ belongs to $\mathcal{L}^{\infty}(0,1)$, so that for the curvature/normal stress boundary condition (2.16) to be at least meaningful, it is necessary to prove that the quantity $\tau_{ij} = S_{ij}(\mathbf{u}(f), p(f))$ (or $\tau_n = \tau_{ij}n_jn_i$) is well-defined on Γ and belongs to $\mathcal{L}^{\infty}(\Gamma)$. This is equivalent to proving that p and $u_{i,j}, i, j = 1, 2$, belong to

$H^1(O)$ with their traces in $\mathcal{L}^\infty(O,1)$, where O is some subdomain of Ω that contains a neighbourhood of Γ.

If we strengthen the condition above slightly by requiring that $O = \Omega$, then it implies (4.24). In the case of problem IV, this is equivalent to

(5.27) $(\eta(f),v)^{00} = \int_0^1 \tau_n S(f)v\, dx \ \ \forall\, v \in N,$

where $S(f)$ is defined as above (5.3).

Since $N = \left\{v \in L^2(0,1) \mid v/\rho^{\frac{1}{2}} \in L^2(0,1), (v(s)-v(t))/(s-t) \in L^2((0,1) \times (0,1))\right\}$, it follows that for every $u \in N$ there exist (not necessarily unique) functions X, Y and Z with $X \in L^2(0,1), \rho^{\frac{1}{2}}Y \in L^2(0,1)$ and $(s-t)Z(s,t) \in L^2((0,1) \times (0,1))$ such that

(5.28) $(u,v)^{00} = \int_0^1\int_0^1 Z(s,t)(v(s)-v(t))\, dsdt \ + \int_0^1 (Y+X)v\, dx \ \ \forall\, v \in N.$

(Take, e.g., $X = u, Y = u/\rho$ and $Z(s,t) = (u(s)-u(t))/(s-t)^2$.) Hence, the regularity condition above implies that for $u = \eta(f)$ we may choose $X \in \mathcal{L}(0,1), Y = 0$ and $Z = 0$ in this decomposition, clearly a strong requirement.

It is apparent from the above that the need to establish regularity results for the solutions of problem (Var) is unavoidable in any attempt to solve problem (FBP). This is confirmed by the literature discussed in chapter 1, in which much effort is devoted to the analysis of the differentiability properties of the weak solution of the auxiliary problem and the derivation of suitable *a priori* estimates. In general the regularity results established in these papers rely on the results of Agmon *et al* (1964) or Solonnikov and Scadilov (1973) for general elliptic systems. However, these theorems invariably require greater regularity from the data than we have assumed thus far. Typically, the regularity theorems for problem (Aux) in situations where there is no contact between Γ and $\partial\Omega \setminus \Gamma$, as in problems I and II, are of the form: if $\mathbf{u}_0 \in C^{2,h}(\Sigma)^2, p_a \in C^{1,h}(\Gamma)$ and $\mathbf{f} \in C^{0,h}(\bar{\Omega})$, then the weak solution $(\mathbf{u},p)$ of the auxiliary problem belongs to $C^{2,h}(\bar{\Omega})^2 \times C^{1,h}(\bar{\Omega})$. (The free boundary is then sought in $C^{3,h}$). Moreover, when there is contact between Γ and $\partial\Omega \setminus \Gamma$, as in problem III and IV, the problem is considerably complicated by the presence of corner points on the boundary of the flow domain, as can be seen in the analysis of Solonnikov (1982).

The undertaking of a similar study was considered to be beyond the scope of this project. We shall only remark that since $\mathbf{f} \in C^\infty(\bar{\Omega})^2$ in problems I - IV, the results given in Ladyzhenskaya (1969) for the case of Dirichlet boundary condition (cf. Theorem 3 in Section 1 of Chapter 2, Theorem 6 in Section 5 of Chapter 5 in Ladyzhenskaya (1969), and pp. 115-116 in the 1963 -edition of this text) suggest that the solution of problem (Var) is smooth in every strictly interior domain and continuous on Γ, excluding neighbourhoods of the corner points, under slightly stronger assumptions than those used here.

We conclude by indicating how (2.16) can be formulated as a fixed point equation in f if it is assumed that the regularity problem has been resolved. We shall again only consider problem IV and shall use the same function spaces as before, but similar ideas apply in the case of the other problems and in smoother settings.

Assume that for each $f \in B = \mathcal{F}(f_0, \varepsilon)$ the solution(s) $(\mathbf{u}, p)$ of problem (Aux) is such that τ_n, as a function of x_1, belongs to $\mathcal{L}^\infty(0,1)$, where we define

$\tau_n = E - q$, with $E = (2/Re)Dij(\mathbf{u})n_j n_i$ on Γ and $p = q + r$ on $\bar{\Omega}$, with $\int_\Omega q \; dx = 0$ and $r \in \Re$ a constant that will be fixed later.

Then (2.16) becomes

$$(1/R)(f)(x) = k(\tau_n(x) - r + p_a(x)), \; x \in [0,1].$$

Assume furthermore that there exists constants U and V, which depend only on h and δ, such that

$$\|\tau_n(f_1) - \tau_n(f_2)\|_\infty \leq UD\|f_1 - f_2\| \;\; \forall\, f_1, f_2 \in B,$$
$$\|\tau_n(f) - \overline{\tau_n(f)}\|_\infty \leq VD \;\; \forall\, f \in B,$$

where $\overline{(\cdot)}$ denotes $\int_0^1 (\cdot) dx$.

Under these assumptions it follows from elementary but lengthy arguments, which we shall omit, that when D and $\|p_a - \overline{p_a}\|$ are sufficiently small, (2.16) is equivalent to the equation $f = F(f)$, where the operator F is a contraction on the complete metric space B and is defined by

$$F(f)(x) = \int_0^x Q(f)(s)ds - \int_0^1 \int_0^t Q(f)(s)\, dsdt$$

where $Q(f) = P(f)/\sqrt{1 - P(f)^2}$ and

$$P(f)(x) = k \int_0^x (\tau_n(f) + p_a)ds - k.r.x - C = k(\int_0^x (\tau_n(f) + p_a)ds - (\overline{\tau_n(f)} + \overline{p_a})x) + (2x-1)C$$

with $r = \overline{\tau_n} + \overline{p_a} - 2C/k, \;\; C = c/\sqrt{1+c^2}$.

Hence, under the assumptions above, F has a unique fixed point f in B, so that problem (FBP) has a solution $(\mathbf{u}(f), p(f), f)$ which is locally unique.

6 References

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